Formale Systeme Proseminar

Tasks for Week 13, 14.1.2016

Task 1 The sequence $(a_i|i \in \mathbb{N})$ is inductively defined by

$$a_0 = 1, \qquad a_{n+1} = \left(\sum_{k=0}^{n} a_k\right) - n + 1.$$

Prove that $\forall n \in \mathbb{N}.a_n = 2^{n-1} + 1$.

Task 2 Prove that $A \subseteq B \Rightarrow |A| \leq |B|$.
(Note: You need to construct an injection from A to B.)

Task 3 Prove that any subset of a finite set is finite.
(Note: you need to show that if A is a finite set, i.e., there is a bijection $f\colon A \to \mathbb{N}_k$ for some $k \in \mathbb{N}$ and $B \subseteq A$, then there is a bijection $g\colon B \to \mathbb{N}_m$ for some $m \in \mathbb{N}$.)

Task 4 Let $A_{m,n} = \{k \in \mathbb{N} \mid n \leq k \leq m\}$. Prove that $A_{m,n}$ is a finite set, by explicitly defining a bijection (to one of the sets $\mathbb{N}_l$ for some $l \in \mathbb{N}$).

Task 5 Prove by induction that if A is a finite set, i.e., $|A| = k$ for some $k \in \mathbb{N}$ then

$$|\mathcal{P}(A)| = 2^k.$$

(Note: We will prove this property in general in class (for arbitrary cardinals), but for finite cardinals (natural numbers), we can prove it concretely using induction and here 2^k is a natural number, the number of elements in $\mathcal{P}(A)$).

Task 6 Prove that for arbitrary cardinals α and β we have $\alpha \cdot \beta = \beta \cdot \alpha$.
(Note: $\alpha = |A|$ for some set A, $\beta = |B|$ for some set B. Also, $\alpha \cdot \beta = |A \times B|$ and $\beta \cdot \alpha = |B \times A|$. So your task here is to give a bijection from $A \times B$ to $B \times A$.)