Formale Systeme Proseminar

Tasks for Week 6

Task 1 Which of the following relations between $A = \{a, b, c\}$ and $B = \{1, 2\}$ define functions from A to B?

(a) $R_1 = \{(a, 1), (b, 2)\}$.
(b) $R_2 = \{(a, 1), (b, 1), (b, 2), (c, 1)\}$.
(c) $R_3 = \{(a, 1), (b, 2), (a, 2)\}$.
(d) $R_4 = \{(a, 1), (b, 2), (c, 1)\}$.

Why?

Task 2 Let $A = \{a, b, c\}$ and $B = \{1, 2\}$. Let $f: A \to B$ be given by

$$f(a) = 1, f(b) = 2, f(c) = 1.$$

(a) Write down the set $f(A')$ for $A' = \{a, c\}$.
(b) Write down the set $f^{-1}(B')$ for $B' = \{2\}$.
(c) Is f injective, surjective, or bijective?
(d) Let $g: B \to B$ be given by $g(1) = g(2) = 2$. Write down the function $g \circ f$.

Task 3 Let $f: \mathbb{Z} \to \mathbb{Z}$ be given by $f(k) = 12 - k$. Show that f is a bijection. Find the inverse function $f^{-1}: \mathbb{Z} \to \mathbb{Z}$ of f.

Task 4 Let $X = \{1, 2, 3, 4, 5\}$ and consider the function $c: \mathcal{P}(X) \setminus \{\emptyset\} \to X$ defined by $c(Y) = |Y|$ for any $Y \subseteq X$, $Y \neq \emptyset$. Show that c is a surjective function, but c is not injective.

Task 5 Let $A = \{1, 2\}$ and $B = \{a, b, c\}$.

(a) Define an injective function $f: A \to B$. For this function f, define a function $f': B \to A$ such that $f' \circ f = id_A$. Is there a surjective function from A to B? Why?
(b) Define a surjective function $g: B \to A$. For this function g, define a function $g': A \to B$ such that $g \circ g' = id_B$. Is there an injective function from B to A? Why?

Task 6 Show that $f: A \to B$ is injective if and only if there exists a function $g: B \to A$ such that $g \circ f = id_A$, i.e., f has a left inverse.

Task 7 Show that $f\colon A \to B$ is surjective if and only if there exists a function $g\colon B \to A$ such that $f \circ g = id_B$, i.e., f has a right inverse.

Task 8 Consider the relation $\sim$ on sets, defined as $A \sim B$ if there exists a bijection $f\colon A \to B$. Prove that $\sim$ is an equivalence relation.