Functions*

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Our topic for this week is functions. Functions are everywhere, from programming to mathematical analysis of algorithms. A function computes output, given an input. Abstractly seen, functions are assignments of a unique element (called image) in a set of outputs to every element in a set of inputs.

Definition 1 (Function, Map, Mapping / D. Funktion, Abbildung). *Let A and B be sets. A function f from A to B, notation $f\colon A \to B$ is an assignment of elements of B to elements of A that satisfies: For every element $a \in A$, there is* exactly one *element $b \in B$ that is assigned to a.*

We write $f(a)$ for the element in B assigned to a, and also write $a \mapsto f(a)$. We say that $f(a)$ is the image (D. Bild) of a, and that a is the original of $f(a)$.

We can write down the requirement that a function must satisfy with the following two predicate logic formulas:

$$\forall a \in A.\, \exists b \in B.\, b = f(a)$$

$$\forall a_1, a_2 \in A.\, a_1 = a_2 \Rightarrow f(a_1) = f(a_2)$$

The second formula needs some more thought: it expresses the uniqueness of the image. Convince yourself (try calculations) that this is indeed the case.

For a function $f\colon A \to B$, the set A is called the *domain* of f (D. Definitionsbereich) and B is called the *codomain* of f (D. Wertebereich). We write $\operatorname{dom} f$ for the domain, and $\operatorname{cod} f$ for the codomain.

Definition 2 (Equality of functions). *Two functions $f\colon A \to B$ and $g\colon C \to D$ are equal iff*

(1) $A = C$, *i.e.,* $\operatorname{dom} f = \operatorname{dom} g$
(2) $B = D$, *i.e.,* $\operatorname{cod} f = \operatorname{cod} g$
(3) $\forall a \in A.\, f(a) = g(a)$.

* Notes from the lectures Formale Systeme on naive set theory. Many thanks to Luis Thiele for helping me with producing the notes.

Definition 3 (Graph). *If $f\colon A \to B$ is a function, then*

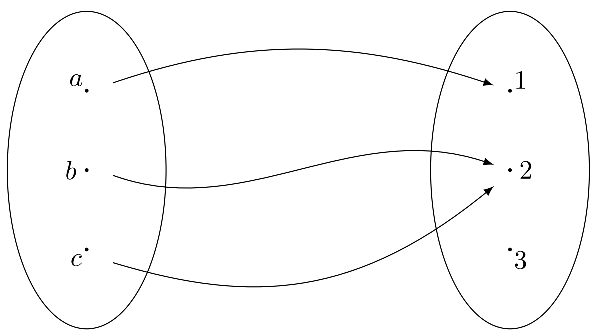
$$\mathrm{graph}(f) = \{(x, y) \in A \times B \mid y = f(x)\}$$

is a relation, called the graph *of f.*

It should be clear that not every relation is the graph of a function.

Example 1. Here are several examples of functions:

(1) Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$ and define $f\colon A \to B$ by $f(a) = 1$, $f(b) = 2$, $f(c) = 2$. Then we can picture f (similarly like we pictured relations) by



We can also define this function f by writing

$$f = \begin{pmatrix} a & b & c \\ 1 & 2 & 2 \end{pmatrix}$$

which is a useful way of presenting functions on a small domain.

(2) By $f(n) = 2n$ for arbitrary $n \in \mathbb{N}$, we have defined a function $f\colon \mathbb{N} \to \mathbb{N}$.

(3) Let $X = \{1, 2, 3\}$ and define $f\colon \mathcal{P}(X) \to \mathcal{P}(X)$ by $f(U) = U \setminus \{1\}$. Concretely:

$$\begin{array}{lcllcl} f(\emptyset) & = & \emptyset & f(\{1,2\}) & = & \{2\} \\ f(\{1\}) & = & \emptyset & f(\{1,3\}) & = & \{3\} \\ f(\{2\}) & = & \{2\} & f(\{2,3\}) & = & \{2,3\} \\ f(\{3\}) & = & \{3\} & f(\{1,2,3\}) & = & \{2,3\} \end{array}$$

(4) For $\mathbb{P}$ denoting the set of all partitions on a set A and $\mathbb{E}$ the set of all equivalence relations on A we have that the assignments $P\colon \mathbb{E} \to \mathbb{P}$ and $R\colon \mathbb{P} \to \mathbb{E}$ given by

$$P(R) = A/R,$$

$$R(P) = \{(x, y) \in A \times A \mid \exists U \in P.\, x \in U \wedge y \in U\}$$

are functions.

1 Image (D. Bild) and Inverse Image (D. Urbild)

The notions of image of an element in the domain and original of an element in the codomain extend naturally to subsets of elements of the domain and codomain, respectively. In this section we show how, and present a small useful property about this image and inverse image.

Definition 4 (Image). *Let $f\colon A \to B$ and $A' \subseteq A$. The Image (D. Bild) of A' is the set*

$$f(A') \stackrel{def}{=} \{f(a) \mid a \in A'\}$$

Clearly $f(A') \subseteq B$. Another way to write the set $f(A')$ is:

$$f(A') = \{b \in B \mid \exists a \in A'.\, b = f(a)\}$$

From the definition, we see that

$$a \in A' \Rightarrow f(a) \in f(A'). \tag{1}$$

The opposite of this implication does not hold (in general), as the next example shows.

Example 2. Let $A = \{a, b, c\}$, $B = \{1, 2, 3\}$, and consider the funciton $f\colon A \to B$ defined by $f(a) = f(b) = 1$, $f(c) = 3$. For $A' = \{a, c\}$, $f(A') = \{f(a), f(c)\} = \{1, 3\}$. Then clearly $f(b) = 1 \in f(A')$, but $b \notin A'$.
For $A' = \{a, b\}$, $f(A') = \{1\}$.

Definition 5 (Inverse Image). *Let $f\colon A \to B$ and $B' \subseteq B$. The inverse image (D. Urbild) of B' is the set*

$$f^{-1}(B') \stackrel{def}{=} \{a \mid f(a) \in B'\}$$

Clearly $f^{-1}(B') \subseteq A$. Here (by the definition of inverse image) we have

$$a \in f^{-1}(B') \Leftrightarrow f(a) \in B'. \tag{2}$$

The inverse image induces a function

$$f^{-1}\colon \mathcal{P}(B) \to \mathcal{P}(A).$$

We can now formulate and prove the following property.

Lemma 1. *Let $f\colon A \to B$, $A' \subseteq A$, $B' \subseteq B$. Then*

$$A' \subseteq f^{-1}(f(A')) \quad \textit{and} \quad f(f^{-1}(B')) \subseteq B'$$

In general, equality does not hold.

Proof. Let $f\colon A \to B$, $A' \subseteq A$, $B' \subseteq B$. In order to prove $A' \subseteq f^{-1}(f(A'))$, let $a \in A'$ be arbitrary. Then $f(a) \in f(A')$ and hence

$$a \in f^{-1}(f(A')).$$

Equality need not hold, as the following example shows: Let $A = \{a, b\}$, $B = \{1\}$,

$$f = \begin{pmatrix} a & b \\ 1 & 1 \end{pmatrix},$$

i.e., $f(a) = f(b) = 1$. Consider $A' = \{a\}$. Then $f(A') = \{1\}$ and $f^{-1}(f(A')) = \{a, b\}$. Obviously $A' \subset f^{-1}(f(A'))$ and hence $A' \neq f^{-1}(f(A'))$. For the other part, for $f(f^{-1}(B')) \subseteq B'$, let $b \in f(f^{-1}(B'))$ Then there exists $a \in f^{-1}(B')$ s.t. $f(a) = b$. Now from $a \in f^{-1}(B')$ we have $f(a) \in B'$. But $f(a) = b$, so $b \in B'$. Since b was arbitrary, we have proven that

$$f(f^{-1}(B')) \subseteq B'.$$

Also here equality does not hold in general, as shown by the next example.
Let $A = \{1\}$, $B = \{a, b\}$, and $f\colon A \to B$ given by $f(1) = a$. Moreover, let $B' = B = \{a, b\}$. We have

$$f^{-1}(B') = \{1\} = A$$

and

$$f(f^{-1}(B')) = f(\{1\}) = \{a\} \subset B'.$$

□

2 Special Functions: Injections, Surjections, Bijections

We have seen that it is possible for a function that (a) two elements in the domain are mapped to the same element in the codomain, or (b) that an element in the codomain is not an image of any element of the domain. We will now exclude such possibilities, and define two special types of functions, one where (a) can not happen, and another one where (b) can not happen.

Definition 6 (Injection). *A function $f\colon A \to B$ is injective, or an injection, iff*

$$\forall a, b \in A.\, f(a) = f(b) \Rightarrow a = b.$$

Definition 7 (Surjection). *A function $f\colon A \to B$ is surjective, or a surjection, iff*

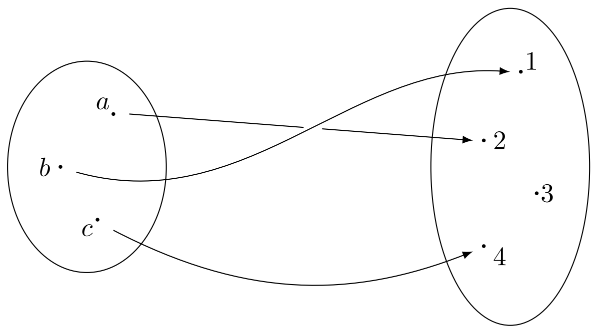
$$\forall b \in B.\, \exists a \in A.\, b = f(a).$$

Definition 8 (Bijection). *A function $f\colon A \to B$ is bijective iff f is both injective and surjective.*

Example 3. Here are some examples of injective and surjective functions.

(1) Let $A = \{a, b, c\}$, $B = \{1, 2, 3, 4\}$. and consider $f\colon A \to B$ defined by

$$f(a) = 2, f(b) = 1, f(c) = 4.$$



This function is injective, as different elements from A have different images. It is not surjective, as 3 is not an image of any element of A.

(2) $f\colon \mathbb{N} \to \mathbb{Z}$ defined by

$$f(n) = -n$$

is injective, too. To show this, let $n, m \in \mathbb{N}$ be such that $f(n) = f(m)$. Then $-n = -m$ and hence $n = m$.

(3) $f\colon \mathbb{Z} \to \mathbb{N}$ defined by

$$f(x) = |x| = \begin{cases} x, & x \in \mathbb{N} \\ -x, & x \in \mathbb{Z} \setminus \mathbb{N} \end{cases}$$

is surjective. For a proof, take arbitrary $n \in \mathbb{N}$. Then $n \in \mathbb{Z}$ and $n = f(n)$.

(4) The function $f\colon \{1, 2, 3\} \to \{a, b, c\}$ given by $f(1) = a$, $f(2) = b$, $f(3) = c$ is bijective.

(5) The function $f\colon \mathbb{N} \to \mathbb{N}$ given by $f(2n) = 2n + 1$, $f(2n + 1) = 2n$ is bijective too.

Moreover, for the functions from Example 1 we have: The function from Example 1(1) is neither injective nor surjective. The function from Example 1(2) is injective but not surjective. The function from Example 1(3) is neither injective nor surjective. Convince yourself in this by providing small proofs! The functions P and R from Example 1(4) are bijective, as implicitly shown by our theorem on partitions and equivalences from last week. Also here you may wish to write down the whole proof as an exercise.

The following lemmas provide some simple characterisations of injectivity, surjectivity, and bijectivity. Note that $|X|$ denotes the cardinality of the set X, a notion that we have not (and could not have!) defined yet but for the purpose here think of $|X| \leq 1$ as "the set X has at most one element", and $|X| \geq 1$ as "the set X has at least one element".

Lemma 2. *A function $f\colon A \to B$ is injective iff*

$$\forall b \in B.\, |f^{-1}(\{b\})| \leq 1$$

Lemma 3. *The following properties are equivalent for a function $f\colon A \to B$:*

1. *f is surjective;*
2. *$\forall b \in B.\, |f^{-1}(\{b\})| \geq 1$;*
3. *$f(A) = B$.*

Surjective functions are also called *onto*, which is justified by the property 3. of Lemma 3, as they map onto the (whole!) codomain set B.

Lemma 4. *A function $f\colon A \to B$ is bijective iff*

$$|f^{-1}(\{b\})| = 1 \textit{ for all } b \in B$$

Let us prove now Lemma 2 as an example, the others you can prove yourself (as homework exercise).

Proof (of Lemma 2). Let $f\colon A \to B$ be injective. Recall that

$$f^{-1}(\{b\}) = \{a \in A \mid f(a) = b\}$$

Assume, towards a contradiction, that there exist two elements $a_1, a_2 \in f^{-1}(\{b\})$ with $a_1 \neq a_2$. But then

$$f(a_1) = b = f(a_2)$$

and hence, since f is injective, $a_1 = a_2$ contradicting the assumption that $a_1 \neq a_2$. This proves that

$$\text{injectivity} \Rightarrow |f^{-1}(\{b\})| \leq 1.$$

For the opposite direction, assume $|f^{-1}(\{b\})| \leq 1$. We will prove that f is injective. Assume $a_1, a_2 \in A$ are such that $f(a_1) = f(a_2)$. Then $a_1 \in f^{-1}(\{b\})$ and $a_2 \in f^{-1}(\{b\})$ for $b = f(a_1) = f(a_2)$. Since $|f^{-1}(\{b\})| \leq 1$ it must be that $a_1 = a_2$, showing that f is injective. $\square$

We can also show two interesting properties that strengthen Lemma 1 under the assumption that the function f is injective or surjective.

Proposition 1. *Let $f\colon A \to B$ be injective and $A' \subseteq A$. Then*

$$f^{-1}(f(A)) = A'.$$

The proof of Proposition 1 relies on the following simple, but useful observation. Note that this is strengthening of the implication in (1) to a bi-implication (equivalence) in case of injective functions.

Lemma 5. *Let $f\colon A \to B$ be injective and let $A' \subseteq A$. Then*

$$f(x) \in f(A') \Leftrightarrow x \in A'.$$

Proposition 2. *Let $f: A \to B$ be surjective and let $B' \subseteq B$. Then*

$$f(f^{-1}(B')) = B'.$$

We will prove Proposition 2 now and leave the proof of Lemma 5 and Proposition 1 as exercise.

Proof (of Proposition 2). Let $f: A \to B$ be surjective and let $B' \subseteq B$. From Lemma 1, we have that it always holds (even without the surjectivity assumption) that

$$f(f^{-1}(B')) \subseteq B' \quad (*)$$

For the remaining inclusion ($\supseteq$) we will need surjectivity. Let $b \in B'$. Since f is surjective, there exists an $a \in A$ with $f(a) = b$. Hence $a \in f^{-1}(B')$. Now $f(a) \in f(f^{-1}(B'))$, but $f(a) = b$ proving that $b \in f(f^{-1}(B'))$. Since b was arbitrary, we have shown

$$B' \subseteq f(f^{-1}(B')) \quad (**)$$

From $(*)$ and $(**)$, $B' = f(f^{-1}(B'))$. □

3 Function Composition and Inverse Function

Let $f: A \to B$ and $g: B \to C$. (Here it is important that $\operatorname{cod} f = \operatorname{dom} g = B$!)

Definition 9 (Composition). *For $f: A \to B$ and $g: B \to C$, the composition $g \circ f$, read "g after f", is the function*

$$g \circ f: A \to C$$

defined by

$$g \circ f(a) = g(f(a))$$

for all $a \in A$.

Lemma 6. *Let $f: A \to B$ and $g: B \to C$ be injective. Then $g \circ f$ is injective.*

Proof. Assume $f: A \to B$ and $g: B \to C$ are injective. Assume $a_1, a_2 \in A$ are such that

$$g \circ f(a_1) = g \circ f(a_2).$$

This means $g(f(a_1)) = g(f(a_2))$. Now since g is injective, it must be that $f(a_1) = f(a_2)$. Since f is injective, from $f(a_1) = f(a_2)$ we conclude $a_1 = a_2$. This proves that $g \circ f$ is injective. □

Lemma 7. *Let $f: A \to B$ and $g: B \to C$ be surjective. Then $g \circ f$ is surjective.*

Proof. Assume f and g are surjective, $f: A \to B$ $g: B \to C$. Let $c \in C'$. Since g is surjective, there exists $b \in B$ such that $g(b) = c$. Further, since $b \in B$ and f is surjective, there exists $a \in A$ with $f(a) = b$. But then

$$c = g(b) = g(f(a)) = g \circ f(a)$$

showing that $g \circ f$ is surjective. □

Corollary 1. *Let $f: A \to B$ and $g: B \to C$ be bijective. Then $g \circ f$ is bijective.* □

Function composition "$\circ$" is a *partial* binary operation[1] on functions. It is binary because it is applied to two arguments which are functions. It is partial because not any two functions can be composed, they have to be compatible (the codomain of the first must be the domain of the second). We now define another important partial unary operation on functions.

Definition 10 (Inverse function). *Let $f: A \to B$ be bijective. Then there exists a function $f^{-1}: B \to A$ read "f inverse", defined by*

$$f^{-1}(b) = a \Leftrightarrow f(a) = b. \tag{3}$$

Lemma 8. *Let $f: A \to B$ bijective. Then*

$$f^{-1} \circ f = id_A$$

$$f \circ f^{-1} = id_B$$

where $id_X: X \to X$ is the function (bijection) defined by

$$id_X(x) = x.$$

Proof. Let $a \in A$. Then $f^{-1}(f(a)) = a$ by (3), since $f(a) = f(a)$ trivially holds. Hence $f^{-1} \circ f = id_A$ as they also have the same domains and codomains, namely the set A.

Let $b \in B$. Then $f(f^{-1}(b)) = b$ by (3) as $f^{-1}(b) = f^{-1}(b)$ trivially holds. Since $f \circ f^{-1}: B \to B$, this proves

$$f \circ f^{-1} = id_B.$$

□

Now we can state and prove an interesting characterisation of bijections. It states that in essence bijections are exactly the functions that have an inverse[2].

Theorem 1. *A function $f: A \to B$ is bijective iff there exists a function $g: B \to A$ with $g \circ f = id_A$ and $f \circ g = id_B$.*

Proof. Let $f: A \to B$ be bijective. Consider the function $g = f^{-1}: B \to A$, which exists for a bijection f. Then by Lemma 8, we have

$$g \circ f = id_A, \quad f \circ g = id_B.$$

For the other direction, assume $f: A \to B$ is a function and there exists a function $g: B \to A$ with

$$g \circ f = id_A, \quad f \circ g = id_B.$$

We need to show that f is bijective.

[1] Note that operations are also functions, of a particular kind. The field of study of operations is *algebra*. I will provide you with a lecture and lecture notes on basics of algebra, too.

[2] In order to understand this comment you might need some knowledge of algebra, which as I already mentioned I will provide you with later on.

Injectivity: Assume $f(x) = f(y)$ for $x, y \in A$. Then $g(f(x)) = g(f(y))$, as g is a function and hence from the assumption that $g \circ f = id_A$ we get

$$x = id_A(x) = g \circ f(x) = g(f(x)) = g(f(y)) = g \circ f(y) = id_A(y) = y.$$

Proving that f is injective.

Surjectivity: For any $b \in B$, we have $g(b) \in A$ and

$$f(g(b)) = f \circ g(b) = id_B(b) = b.$$

$\square$

Our next topic will be *cardinals*. The notion of a cardinal is there to measure the size of a set. Without revealing much upfront, let us briefly discuss the following points:

- Let $A = \{1, 2, 3\}$ and B a set. How many elements must B have at least, so that there exists an injective function $f\colon A \to B$?
  Since all images $f(1)$, $f(2)$, $f(3)$ must be different, B must have at least 3 different elements. So we see that in order that there exists an injection from A to B, B must be at least as big as A. This intuition is generalized to compare the size of any two sets.

- Similarly, in order that a surjection $f\colon A \to B$ exists, A must be at least as big as B, as otherwise we would not have enough arrows to hit all elements of B.

- This leads us to a conclusion that there exists a bijection $f\colon A \to B$ iff A and B are *of equal size*. This we will take later as the definition of ”equal cardinality”.