Back to
Naive Set Theory
Relations

Product of multiple sets

Direct product (Kartesisches Produkt)

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ordered pairs

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In general, for sets $A_1, A_2, \ldots, A_n$ with $n \geq 1$,

$$A_1 \times A_2 \times \ldots \times A_n = \prod_{1 \leq i \leq n} A_i = \{(x_1, x_2, \ldots, x_n) \mid x_i \in A_i \text{ for } 1 \leq i \leq n\}$$

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if $A_i = A$ for all i, then the product is denoted A^n

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similarly, unary relation (subset), n-ary relation...

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Special relations

A relation $R \subseteq A \times A$ is:

reflexive	iff	for all $a \in A$, $(a,a) \in R$
symmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \in R$
transitive	iff	for all $a,b,c \in A$, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$
irreflexive	iff	for all $a \in A$, $(a,a) \notin R$
antisymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$
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(infix) notation aRb for $(a,b) \in R$

Special relations

A relation R on A, i.e., R $\subseteq$ A x A is:

equivalence	iff	R is reflexive, symmetric, and transitive
partial order	iff	R is reflexive, antisymmetric, and transitive
strict order	iff	R is irreflexive and transitive
preorder	iff	R is reflexive and transitive
total (linear) order	iff	R is a total partial order

Obvious properties

1. Every partial order is a preorder.

2. Every total order is a partial order.

3. Every total order is a preorder.

4. If $R \subseteq A \times A$ is a relation such that there are $a, b \in A$ with

   $$a \neq b, (a,b) \in R \text{ and } (b,a) \in R,$$

   then R is not a partial order, nor a total order, nor a strict order.

Operations on relations

Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be two relations. Their composition is the relation

$R \circ S = \{(a,c) \in A \times C \mid \text{there is } b \in B \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\}$

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Let $R \subseteq A \times B$ be a relation. The inverse relation of R is the relation

$$R^{-1} = \{(b,a) \in B \times A \mid (a, b) \in R\}$$

Characterizations

Lemma: Let R be a relation over the set A. Then

1. R is reflexive iff $\Delta_A \subseteq R$
2. R is symmetric iff $R \subseteq R^{-1}$
3. R is transitive iff $R^2 \subseteq R$

Important equivalence on $\mathbb{Z}$

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Lemma: The relation $\equiv_n$ is an equivalence for every n.

Equivalences classes

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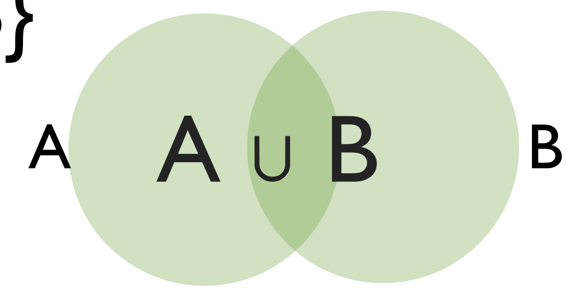
Task: Describe the equivalence classes of $\equiv_n$
How many classes are there?

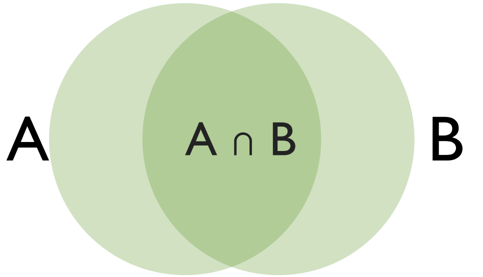
Unions and intersections of multiple sets

Union (Vereinigung) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



Intersection (Durchschnitt) $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

A and B are disjoint if $A \cap B = \varnothing$



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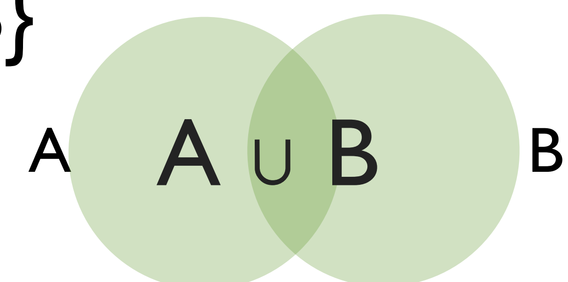

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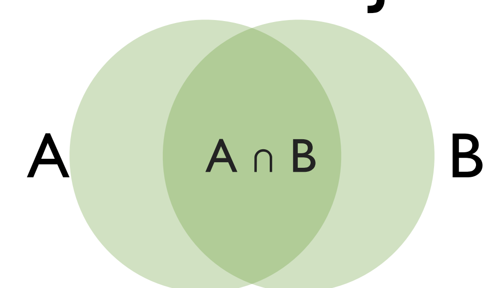

In general, for sets $A_1, A_2, \ldots, A_n$ with $n \geq 1$,

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{1 \leq i \leq n} A_i = \{x \mid x \in A_i \text{ for some } i \in \{1,..n\}\}$$

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{1 \leq i \leq n} A_i = \{x \mid x \in A_i \text{ for all } i \in \{1,..n\}\}$$

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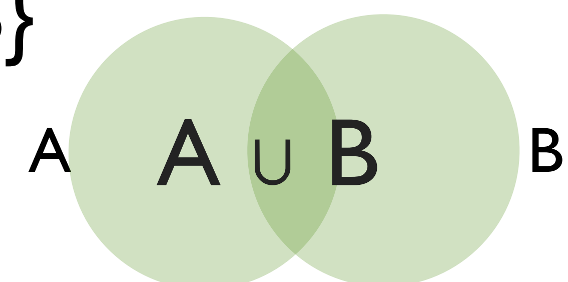

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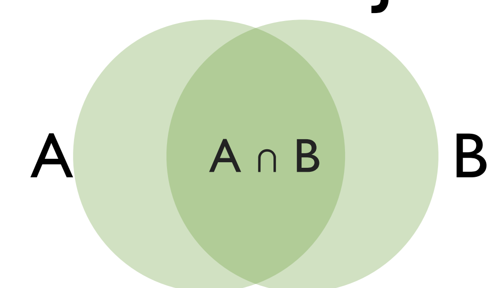

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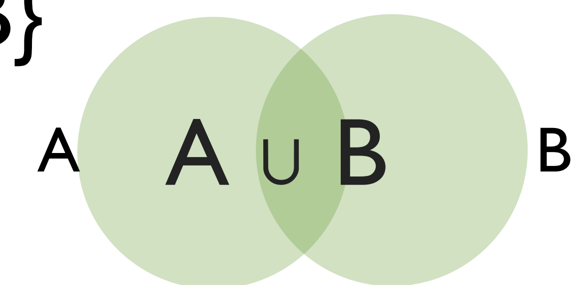

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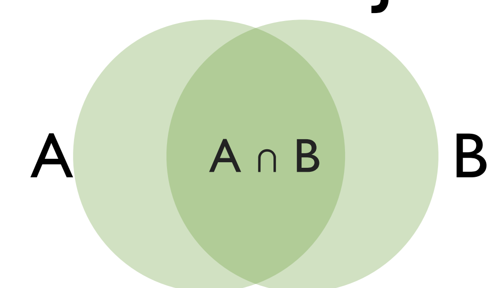

In general, for a family of sets $(A_i \mid i \in I)$

$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$

$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$

Back to equivalence classes

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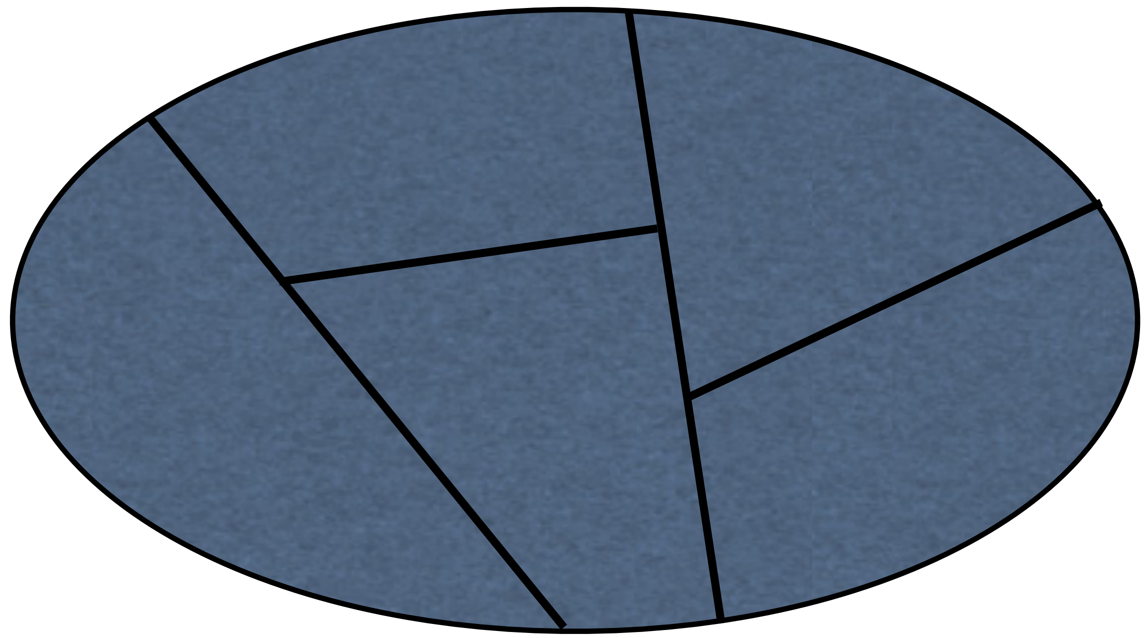
Lemma E2: $A = \bigcup_{a \in A} [a]_R$. The union is disjoint.

Partitions



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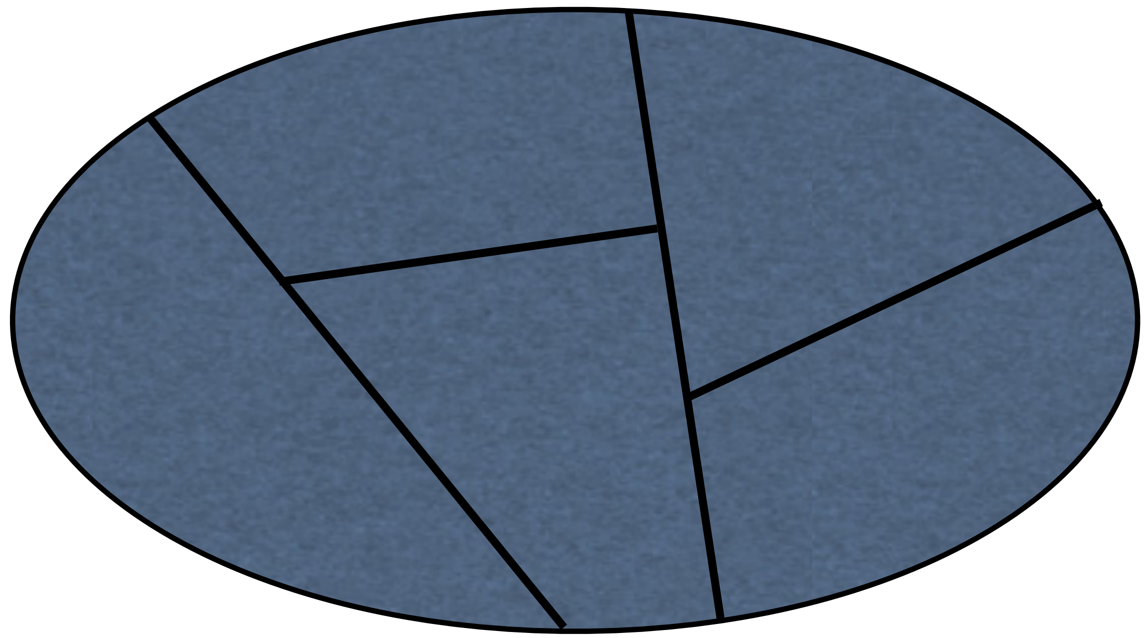

Def. Let X be a set. A subset P of the powerset $\mathcal{P}(X)$ is a partition (Klasseneinteilung) of X if it satisfies:

(1) For all $A \in P$, $A \neq \emptyset$

(2) For all $A, B \in P$, if $A \neq B$ then $A \cap B = \emptyset$

(3) $\bigcup_{A \in P} A = X$

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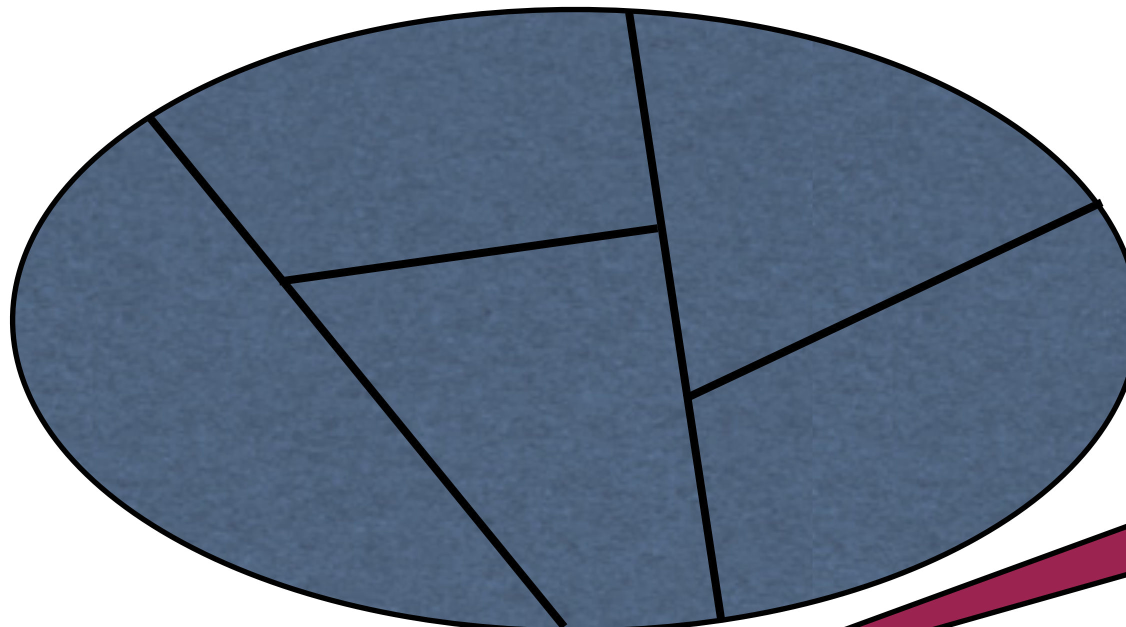

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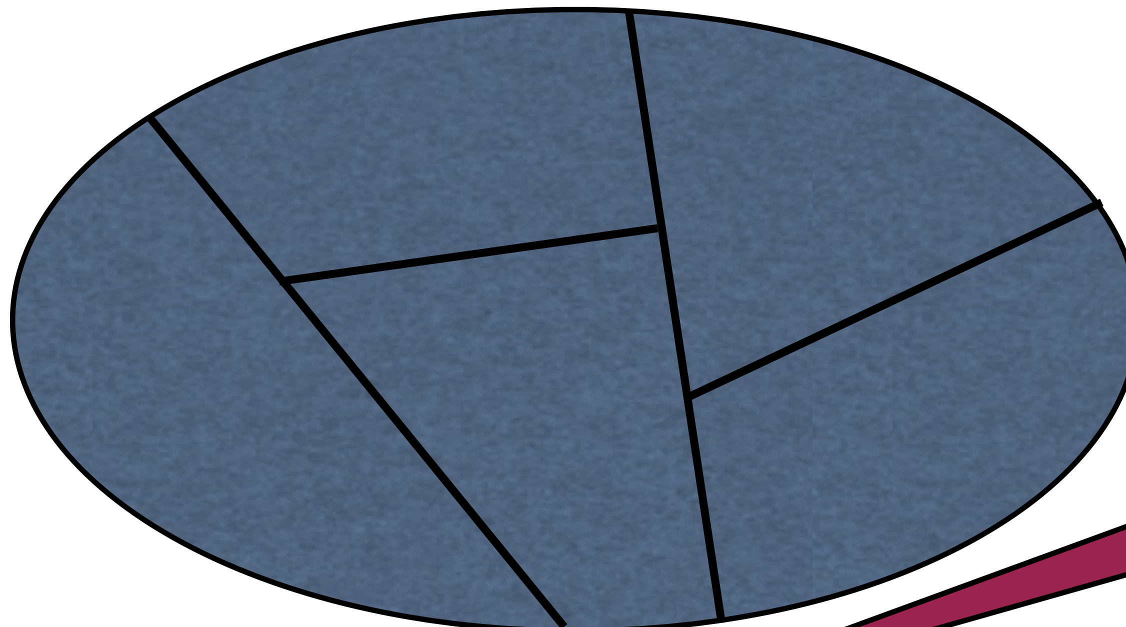

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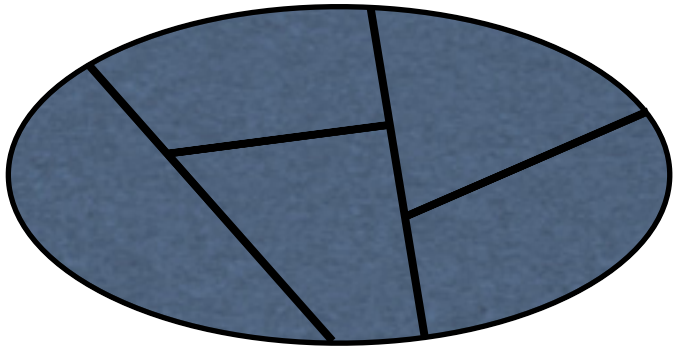
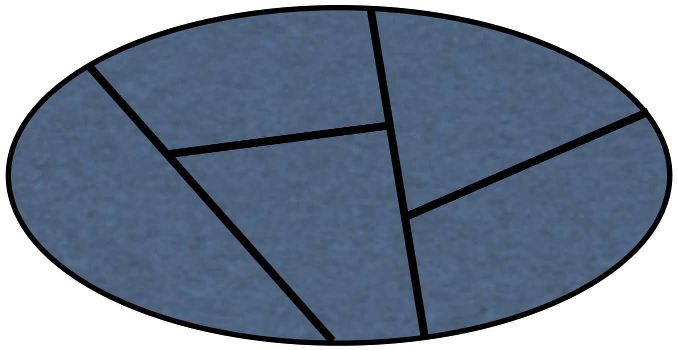
that are non-empty, pairwise disjoint, and their union equals X



Partitions = Equivalences



Partitions = Equivalences

Theorem PE: Let X be a set.

(1) If R is an equivalence on X, then the set

$$P(R) = \{ [x]_R \mid x \in X \}$$

is a partition of X.

(2) If P is a partition of X, then the relation

$$R(P) = \{(x,y) \in X \times X \mid \text{there is } A \in P \text{ such that } x,y \in A\}$$

is an equivalence relation.

Moreover, the assignments $R \mapsto P(R)$ and $P \mapsto R(P)$ are inverse to each other, i.e., $R(P(R)) = R$ and $P(R(P)) = P$.

Transitive closure

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$R^0 = \Delta_R$

Proposition TC: Let R be a relation on X. The transitive closure of R is the smallest transitive relation that contains R. The reflexive and transitive closure of R is the smallest reflexive and transitive relation that contains R.