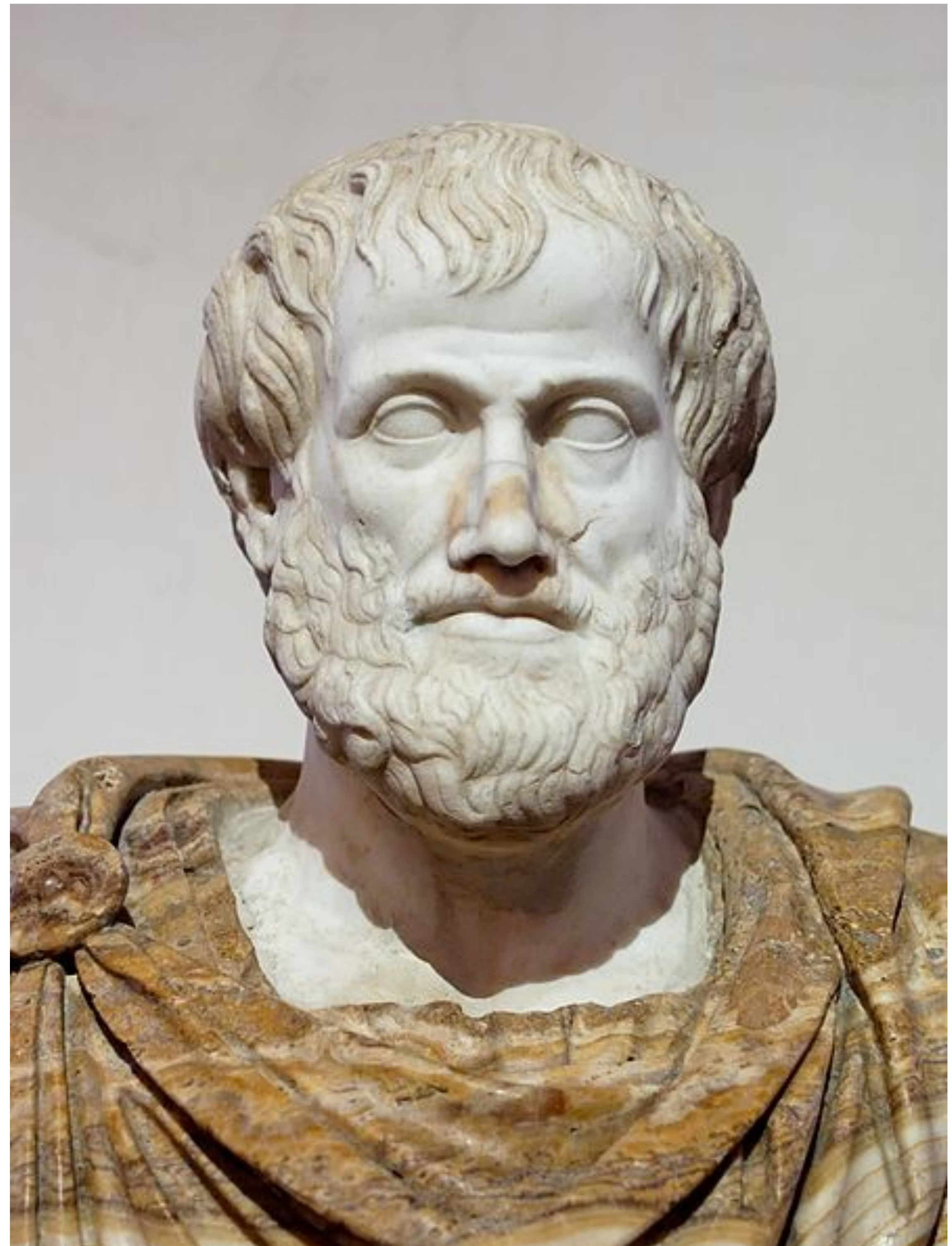
Logic

In the beginning

Aristotle +/- 350 B.C.

Organon

19 syllogisms



Logic = study of correct reasoning

In the beginning

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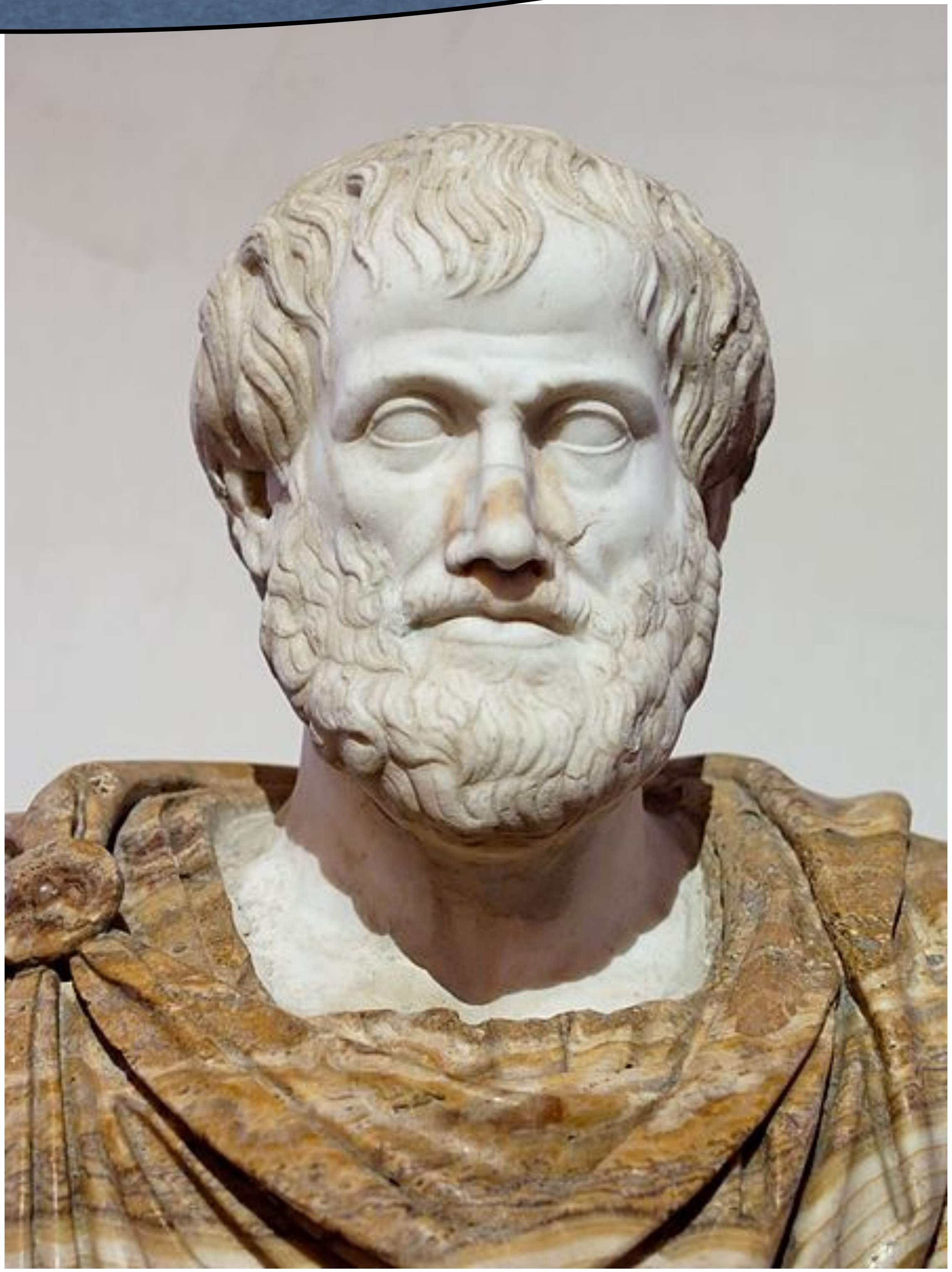

Barbara syllogism

All K's are L's
All L's are M's

All K's are M's

Barbara syllogism

only later called so, in the Middle Ages

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from the two premises

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from the two premises

one can **always** conclude the conclusion

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from the two premises

one can **always** conclude the conclusion

independent of what the parameters K,L,M are

Barbara syllogism

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All K's are L's
All L's are M's

All K's are M's

from the two premises

one can **always** conclude the conclusion

independent of what the parameters K,L,M are

Logic (Logos, Greek for word, understanding, reason) deals with general reasoning laws in the form of formulas with parameters.

Propositions

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Def. A proposition (Aussage) is a grammatically correct sentence that is either true or false.

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logic deals with patterns! what matters are not particular propositions but the way in which (abstract) propositions are combined and what follows from them

Connectives

$\wedge$ for "and"

$\vee$ for "or"

$\neg$ for "not"

$\Rightarrow$ for "if .. then" or "implies"

$\Leftrightarrow$ for "if and only if"

Abstract propositions

Abstract propositions

Definition

Basis	Propositional variables are abstract propositions.
Step (Case 1)	If P is an abstract proposition, then so is $(\neg P)$.
Step (Case 2)	If P and Q are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

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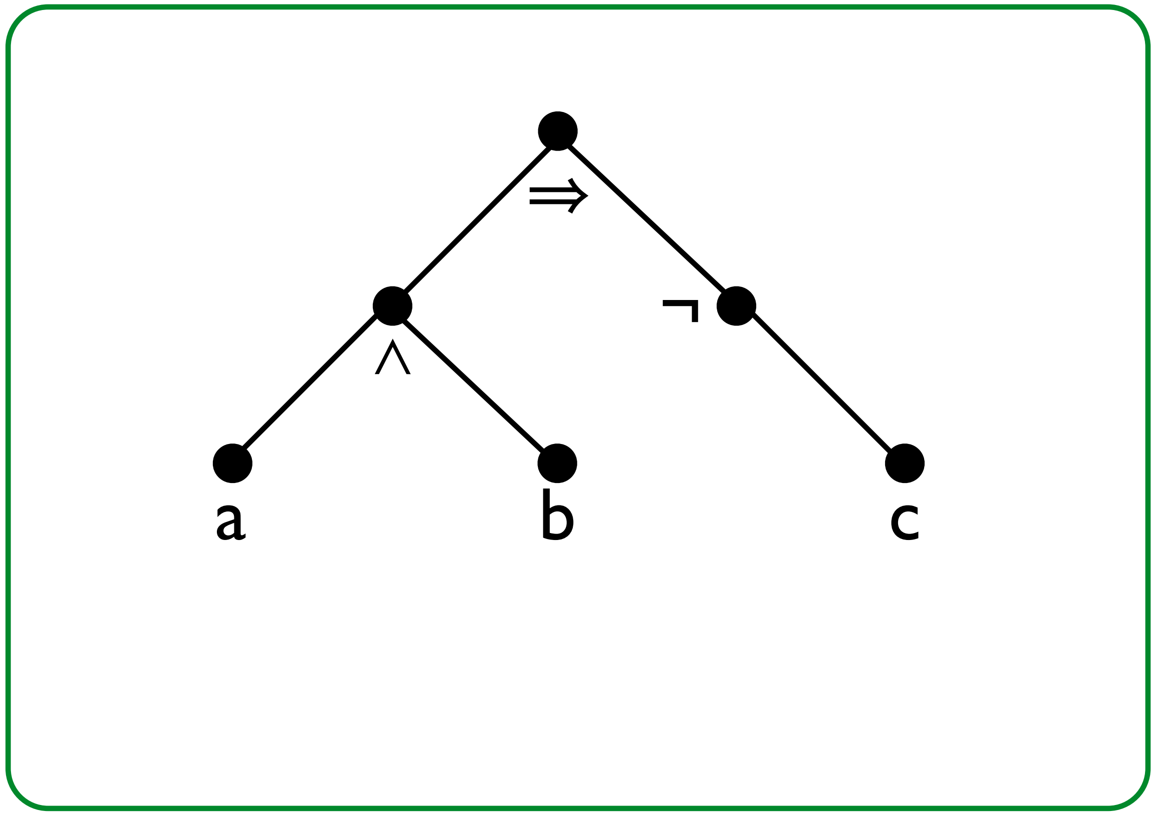
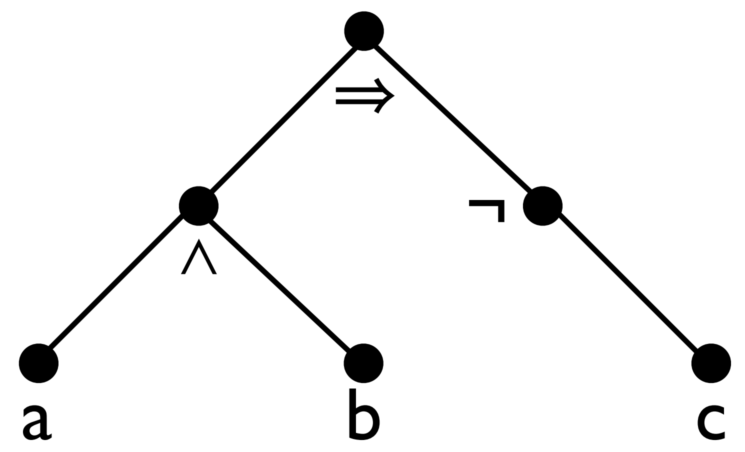
a recursive/inductive definition

…and their structure

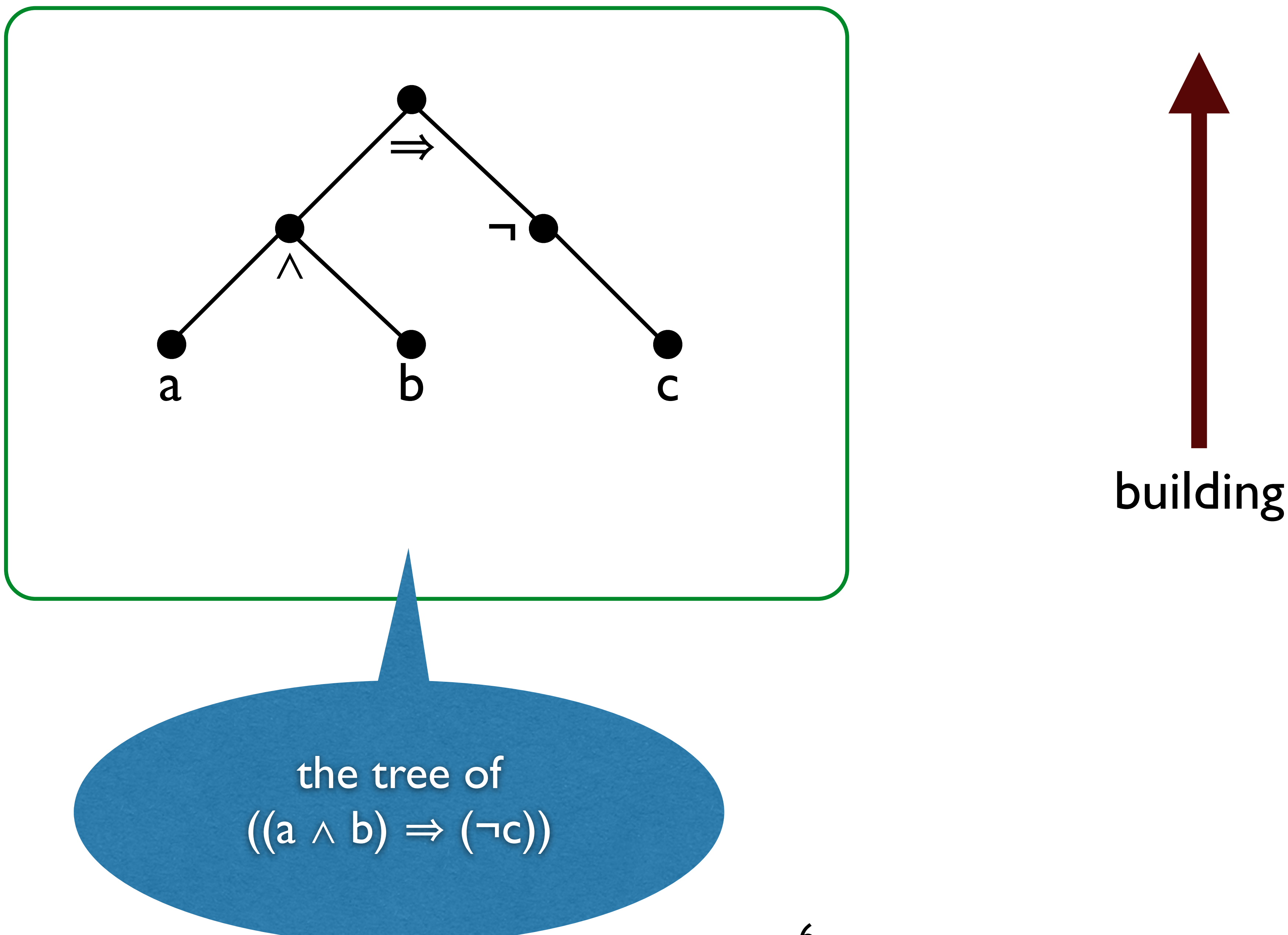


...and their structure



the tree of
$((a \wedge b) \Rightarrow (\neg c))$

...and their structure



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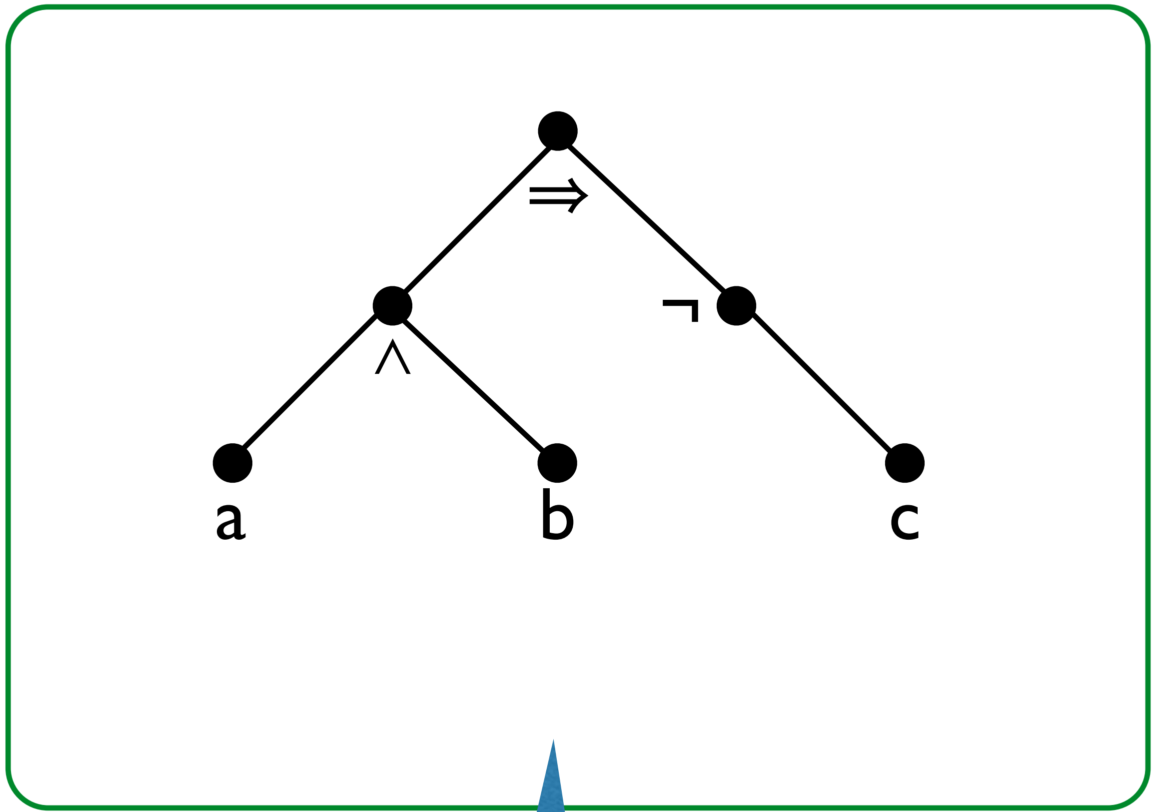

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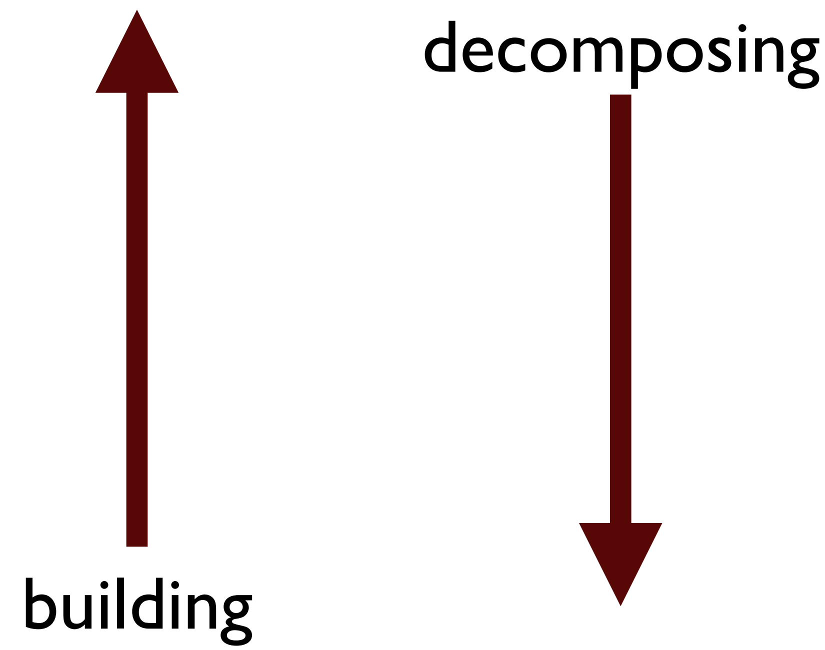
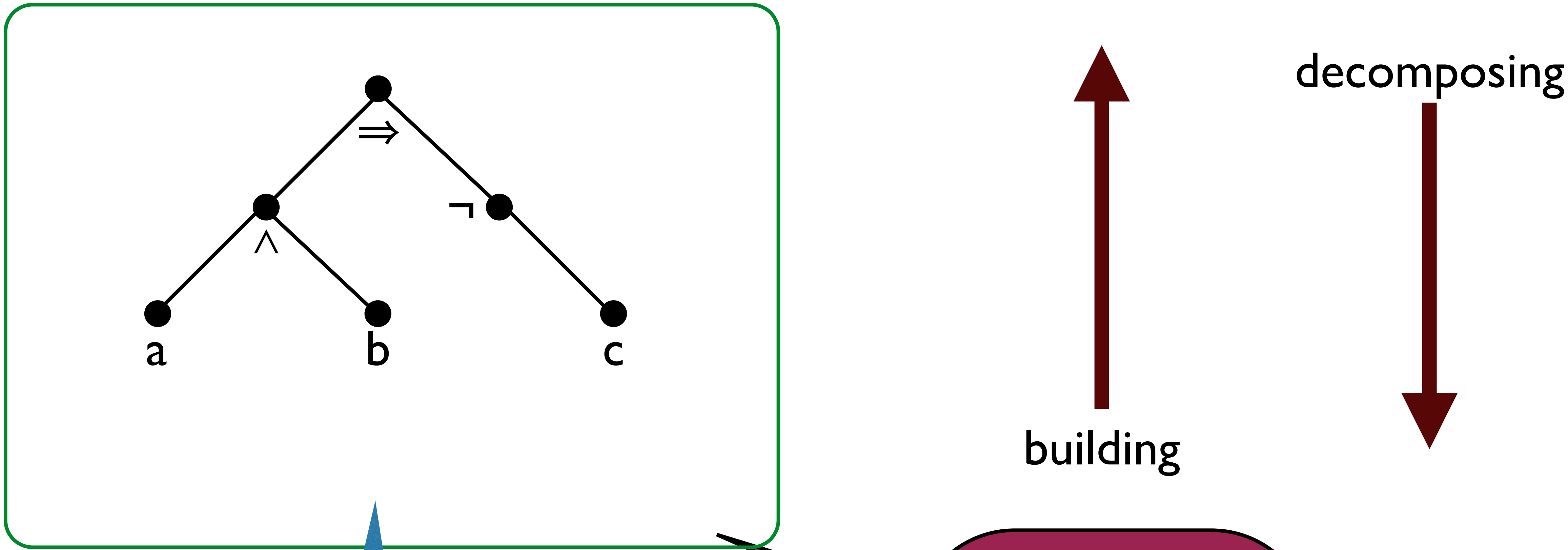


...and their structure



the tree of
$((a \wedge b) \Rightarrow (\neg c))$

tree representation
(no need of
parenthesis)

Dropping parenthesis

$\neg$
$\wedge \quad \vee$
$\Rightarrow$
$\Leftrightarrow$

Dropping parenthesis

$\neg$
$\wedge \quad \vee$
$\Rightarrow$
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priority schema
(top binds the most)

Dropping parenthesis

$\neg$

$\wedge \quad \vee$

$\Rightarrow$

$\Leftrightarrow$

increasing priority

priority schema
(top binds the most)

Dropping parenthesis

$\neg$

$\wedge \quad \vee$

$\Rightarrow$

$\Leftrightarrow$

priority schema
(top binds the most)

decreasing priority

increasing priority

Dropping parenthesis

$\neg$

$\wedge \quad \vee$

$\Rightarrow$

$\Leftrightarrow$

priority schema
(top binds the most)

decreasing priority

increasing priority

Example: $((a \wedge b) \Rightarrow (\neg c))$

becomes

$a \wedge b \Rightarrow \neg c$

Truth tables

Conjunction

P	Q	P∧Q
0	0	0
0	1	0
1	0	0
1	1	1

Truth tables

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P	Q	P∧Q
0	0	0
0	1	0
1	0	0
1	1	1

Truth tables

Conjunction

P	Q	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

only true when both P and Q are true

Truth tables

Disjunction

P	Q	P∨Q
0	0	0
0	1	1
1	0	1
1	1	1

Truth tables

Disjunction

P	Q	P∨Q
0	0	0
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Truth tables

Disjunction

P	Q	P∨Q
0	0	0
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true when either P or Q or both are true

Truth tables

Negation

Truth tables

Negation

unary connective

Truth tables

Negation

unary connective

P	¬P
0	1
1	0

Truth tables

Negation

unary connective

P	¬P
0	1
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Truth tables

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unary connective

P	¬P
0	1
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true when P is false

Truth tables

Implication

Truth tables

Implication

needs more attention

Truth tables

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P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

Truth tables

Implication

needs more attention

P	Q	$P \Rightarrow Q$
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0	1	1
1	0	0
1	1	1

Truth tables

Implication

needs more attention

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
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only false when P is true and Q is false

Truth tables

Bi-implication

Truth tables

Bi-implication

$P \Leftrightarrow Q$

is $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Truth tables

Bi-implication

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is $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
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Truth tables

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Truth tables

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0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

true when P and Q have the same truth value

Truth-functions

Def. A truth-function or Boolean function is a function

$$f: \{0,1\}^n \longrightarrow \{0,1\}$$

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Notation in the book…

$$\begin{array}{l} \quad a, b \\ \left\{ \begin{array}{l} (0,0) \longmapsto 0 \\ (0,1) \longmapsto 1 \\ (1,0) \longmapsto 0 \\ (1,1) \longmapsto 1 \end{array} \right. \end{array}$$

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$P(a,b)$: $(a \wedge b) \vee b$

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$a_1, .. a_n$ are the variables in P (and more) ordered in a sequence

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Truth-functions

$a_1, .. a_n$ are the variables in P (and more) ordered in a sequence

Property: Every abstract proposition $P(a_1,..,a_n)$ with ordered and specified variables induces a truth-function.

Note:

The sequence of specified variables matters!

P(a,b,c): $(a \wedge b) \vee b$

induces

$$
\begin{array}{l}
\quad a, b, c \\
(0,0,0) \longmapsto 0 \\
(0,0,1) \longmapsto 0 \\
(0,1,0) \longmapsto 1 \\
(0,1,1) \longmapsto 1 \\
(1,0,0) \longmapsto 0 \\
(1,0,1) \longmapsto 0 \\
(1,1,0) \longmapsto 1 \\
(1,1,1) \longmapsto 1
\end{array}
$$

Equivalence of propositions

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Definition: Two abstract propositions P and Q are equivalent, notation $P \overset{val}{=} Q$, iff they induce the same truth-function

on any sequence containing their common variables

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Property: The relation $\overset{\text{val}}{=}$ is an equivalence on the set of all abstract propositions

i.e., for all abstract propositions P, Q, R,
(1) $P \overset{\text{val}}{=} P$; (2) if $P \overset{\text{val}}{=} Q$, then $Q \overset{\text{val}}{=} P$; and
(3) if $P \overset{\text{val}}{=} Q$ and $Q \overset{\text{val}}{=} R$, then $P \overset{\text{val}}{=} R$

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0				
0	1				
1	0				
1	1				

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Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
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1	0	0	1	0	0
1	1	0	0	0	0

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0	0	1	1	0	0
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

Their truth values are the same, so they are equivalent

$$b \wedge \neg b \overset{val}{=} c \wedge \neg c$$

Tautologies and contradictions

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Def. An abstract proposition P is a contingency iff
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all contradictions are equivalent

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Tautologies and contradictions

Def. An abstract proposition P is a tautology iff its truth-function is constant 1.

all tautologies are equivalent

Def. An abstract proposition P is a contradiction iff its truth-function is constant 0.

all contradictions are equivalent

but not all contingencies!

Def. An abstract proposition P is a contingency iff it is neither a tautology nor a contradiction.

Abstract propositions

Definition

Basis (Case 1) T and F are abstract propositions.
Basis (Case 2) Propositional variables are abstract propositions.

Step (Case 1) If P is an abstract proposition, then so is $(\neg P)$.
Step (Case 2) If P and Q are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

a recursive/inductive definition

Propositional Logic
Standard Equivalences

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$
$$P \vee Q \stackrel{val}{=} Q \vee P$$
$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

$$P \Rightarrow Q \stackrel{val}{\neq} Q \Rightarrow P$$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0	1	1	0

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

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Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$
$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

Commutativity and Associativity

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$$P \wedge Q \overset{val}{=} Q \wedge P$$
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$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$
$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

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$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0		

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$
$$P \vee Q \overset{val}{=} Q \vee P$$
$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$
$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0	0	1

Idempotence and Double Negation

Idempotence

$$P \wedge P \stackrel{val}{=} P$$

$$P \vee P \stackrel{val}{=} P$$

$$P \Rightarrow P \stackrel{val}{\neq} P$$

$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Idempotence and Double Negation

Idempotence

$$P \wedge P \stackrel{val}{=} P$$

$$P \vee P \stackrel{val}{=} P$$

$$P \Rightarrow P \stackrel{val}{\neq} P$$

$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Double negation

$$\neg\neg P \stackrel{val}{=} P$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

T/F - elimination

$$P \wedge T \stackrel{val}{=}$$

$$P \wedge F \stackrel{val}{=}$$

$$P \vee T \stackrel{val}{=}$$

$$P \vee F \stackrel{val}{=}$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$
$$\neg F \stackrel{val}{=} T$$

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$$P \vee \neg P \stackrel{val}{=} T$$

T/F - elimination

$$P \wedge T \stackrel{val}{=} P$$
$$P \wedge F \stackrel{val}{=} F$$
$$P \vee T \stackrel{val}{=} T$$
$$P \vee F \stackrel{val}{=} P$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \overset{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \overset{val}{=} (P \vee Q) \wedge (P \vee R)$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \overset{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \overset{val}{=} (P \vee Q) \wedge (P \vee R)$$

De Morgan

$$\neg(P \wedge Q) \overset{val}{=} \neg P \vee \neg Q$$

$$\neg(P \vee Q) \overset{val}{=} \neg P \wedge \neg Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \overset{val}{=} \neg P \lor Q$$

$$P \lor Q \overset{val}{=} \neg P \Rightarrow Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \overset{val}{=} \neg P \vee Q$$

$$P \vee Q \overset{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \overset{val}{=} \neg Q \Rightarrow \neg P$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \overset{val}{=} \neg P \vee Q$$

$$P \vee Q \overset{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \overset{val}{=} \neg Q \Rightarrow \neg P$$

$$P \Rightarrow Q \overset{val}{\neq} \neg P \Rightarrow \neg Q$$

common mistake!

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \overset{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \overset{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \overset{val}{=}$$

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Self-equivalence

$$P \Leftrightarrow P \overset{val}{=} T$$

Calculating with equivalent propositions

(the use of standard equivalences)

Recall...

Definition: Two abstract propositions P and Q are equivalent, notation $P \overset{val}{=} Q$, iff they induce the same truth-function

on any sequence containing their common variables

Property: The relation $\overset{val}{=}$ is an equivalence on the set of all abstract propositions.

i.e., for all abstract propositions P, Q, R,
(1) $P \overset{val}{=} P$; (2) if $P \overset{val}{=} Q$, then $Q \overset{val}{=} P$; and
(3) if $P \overset{val}{=} Q$ and $Q \overset{val}{=} R$, then $P \overset{val}{=} R$

Substitution

Simple

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]}$$

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EVERY occurrence of P is substituted!

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Simultaneous

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{=} \psi[\xi/P, \eta/Q]}$$

Substitution

meta rule

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The rule of Leibnitz

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formula that has ϕ as a sub formula

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Strengthening and weakening

We had

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{val}{=} Q$, iff

(1) Always when P has truth value 1, also Q has truth value 1, and

(2) Always when Q has truth value 1, also P has truth value 1.

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in an equivalent formulation

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Strengthening

Definition: The abstract proposition P is stronger than Q, notation $P \overset{val}{\models} Q$, iff

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Lemma W4: $P \overset{val}{\models} Q$ iff $P \Rightarrow Q$ is a tautology.

Standard Weakenings

and-or-weakening

$$P \wedge Q \overset{val}{\models} P$$

$$P \overset{val}{\models} P \vee Q$$

Extremes

$$\mathrm{F} \overset{val}{\models} P$$

$$P \overset{val}{\models} \mathrm{T}$$

Calculating with weakenings
(the use of standard weakenings)

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does not hold for weakening!

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Monotonicity

$$\frac{P \overset{val}{\models} Q}{P \wedge R \overset{val}{\models} Q \wedge R}$$

$$\frac{P \overset{val}{\models} Q}{P \vee R \overset{val}{\models} Q \vee R}$$