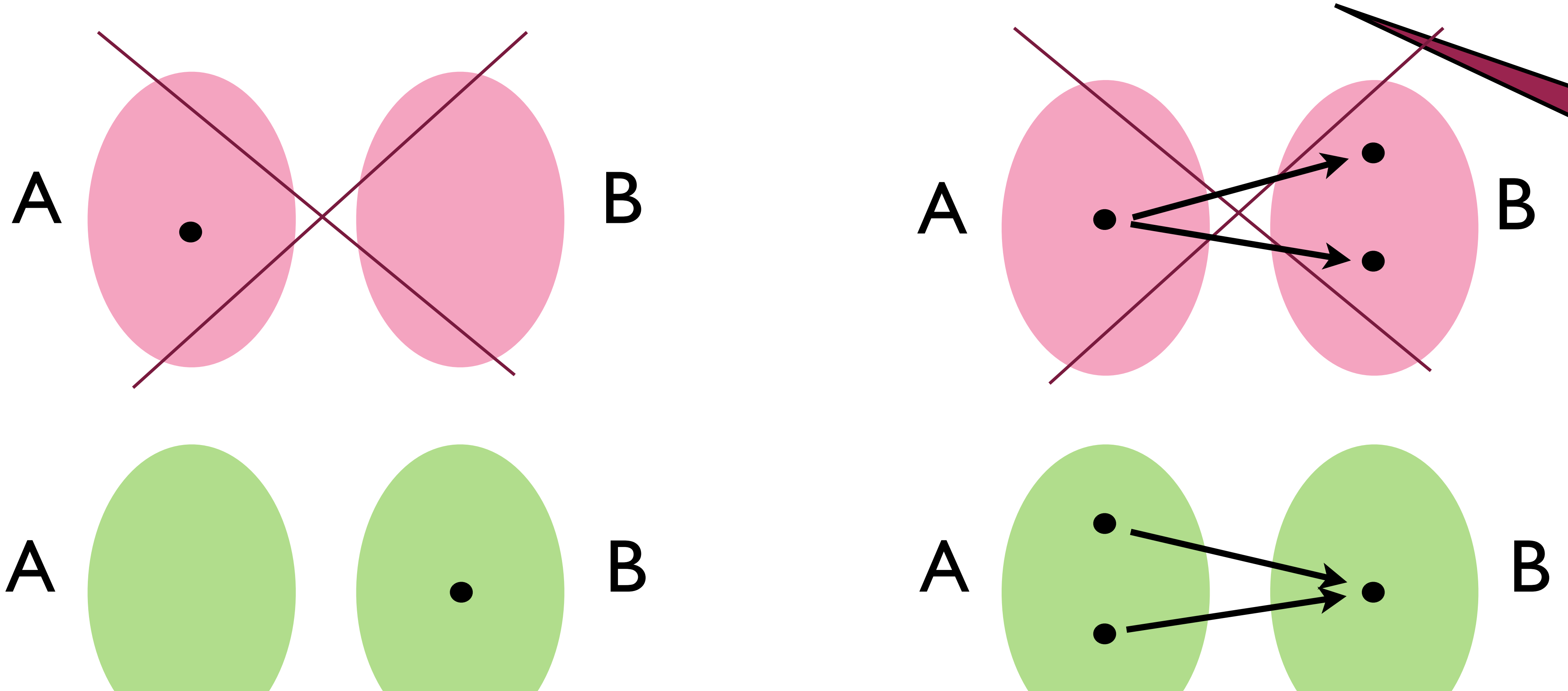
Functions, mappings

Def. If A and B are sets, a function (mapping, Abbildung) f from A to B, notation f: $A \longrightarrow B$ is an assignment (of elements of B to elements of A, we write f(a) for the element assigned to a) s. t.

for every $a \in A$, there exists a unique $b \in B$ such that $b = f(a)$.



the image of a,
$b = f(a)$
$a \mapsto b$

$\{(a, f(a)) \mid a \in A\}$ is the graph of the function f

Functions, mappings

When $f\colon A \longrightarrow B$ then $\mathrm{dom}\, f = A$ and $\mathrm{cod}\, f = B$

domain of F
(Definitionsbereich)

codomain of F
(Wertebereich)

Equality of functions

Let $f: A \longrightarrow B$ and $g: C \longrightarrow D$

Def. The functions $f: A \longrightarrow B$ and $g: C \longrightarrow D$ are equal iff

(1) $A = C$

(2) $B = D$

(3) for all $a \in A$, $f(a) = g(a)$.

dom f = dom g

cod f = cod g

Image

Let f: $A \longrightarrow B$ and $A' \subseteq A$.

The image (Bild) of A' is the set $f(A') = \{f(a) \mid a \in A'\} \subseteq B$.

$f(A') = \{b \in B \mid \text{there is an } a \in A' \text{ with } b = f(a)\}$

if $a \in A'$, then $f(a) \in f(A')$

So f extends to a function f: $\mathcal{P}(A) \longrightarrow \mathcal{P}(B)$, the image-function.

Inverse image

Let f: $A \longrightarrow B$ and $B' \subseteq B$.

The inverse image (Urbild) of B' is the set

$$f^{-1}(B') = \{a \mid f(a) \in B'\} \subseteq A.$$

$a \in f^{-1}(B')$ iff $f(a) \in B'$

Again the inverse image induces a function $f^{-1}: \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$, the inverse-image-function.

Lemma F1: Let f: $A \longrightarrow B$, $A' \subseteq A$, and $B' \subseteq B$. Then

$$A' \subseteq f^{-1}(f(A')) \quad \text{and} \quad f(f^{-1}(B')) \subseteq B'$$

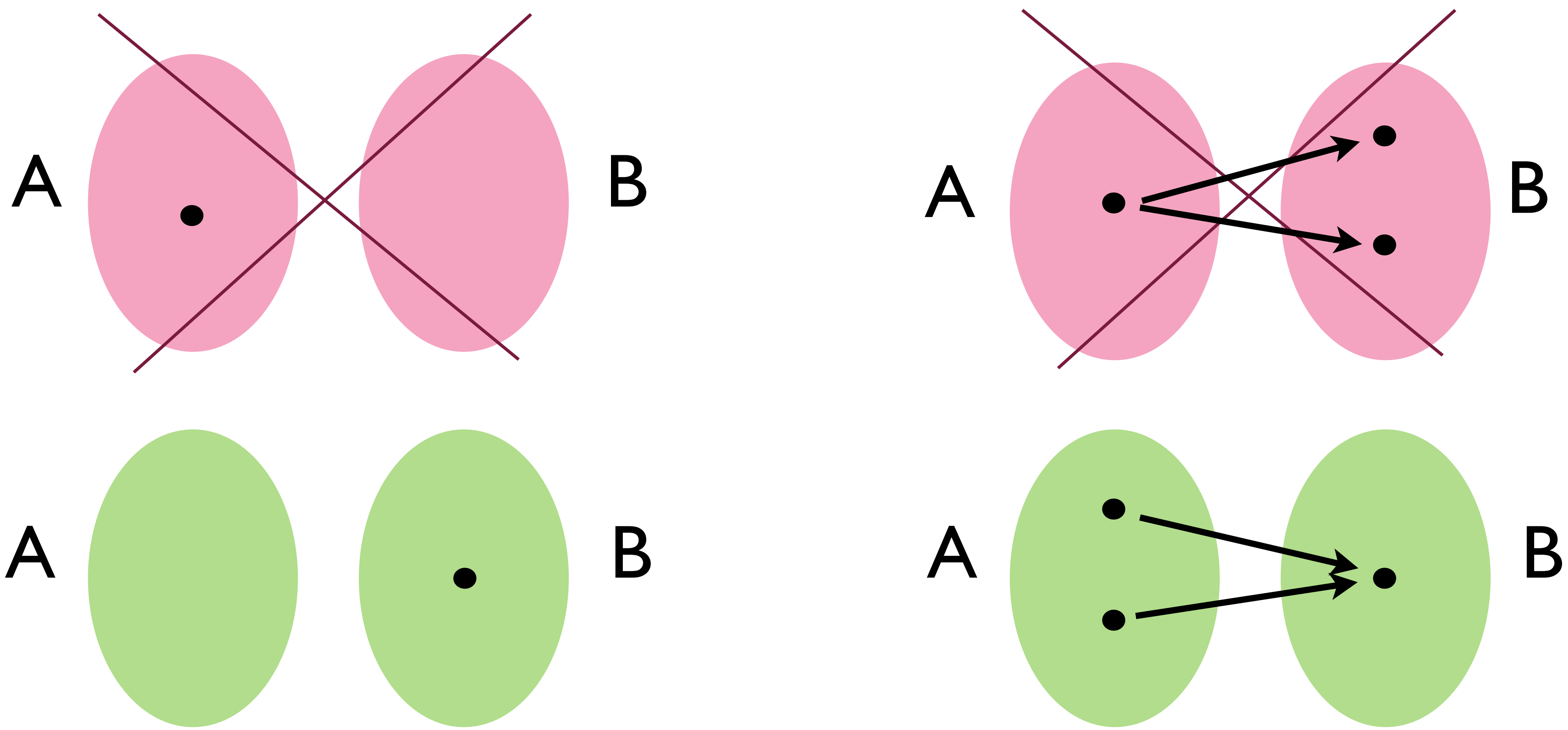
(in general no more than this holds)

Recall…

Def. If A and B are sets, a function f from A to B, notation

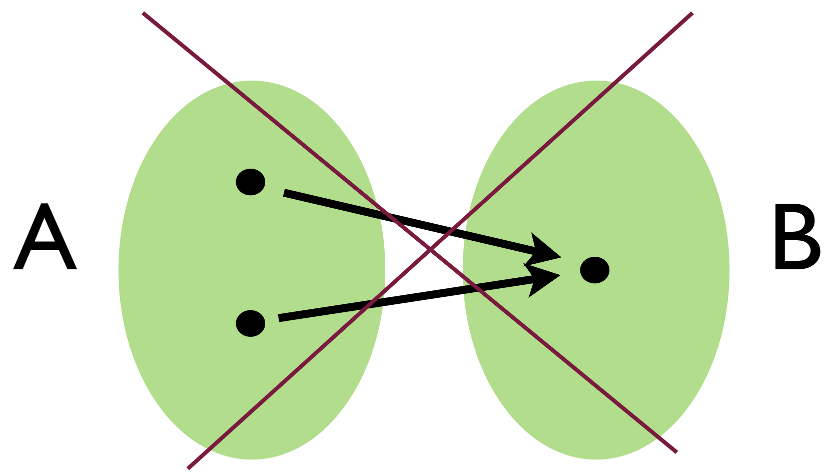
$f: A \longrightarrow B$ is an assignment s. t.

for every $a \in A$, there exists a unique $b \in B$ such that $b = f(a)$.

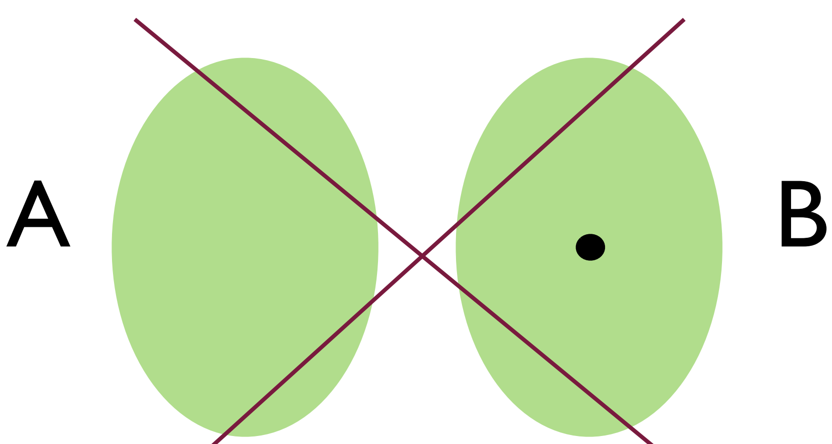


Special functions

The number of ingoing arrows for a function can be 0,1, or more.
Based on this, we distinguish some special functions.



at most one incoming arrow
injection



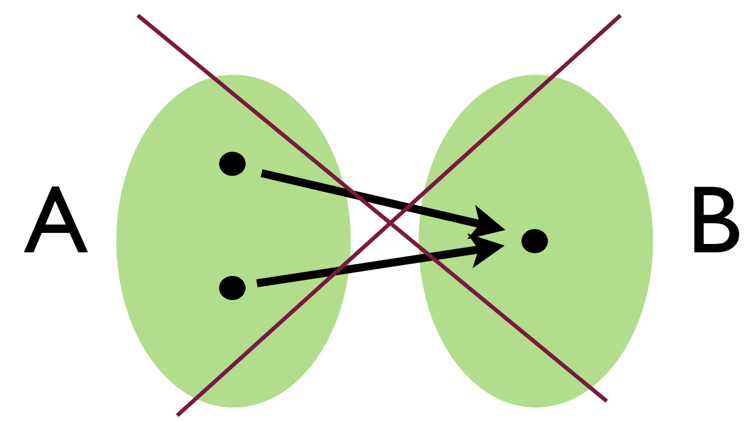
at least one incoming arrow
surjection

exactly one incoming arrow (injection + surjection) bijection

Special functions

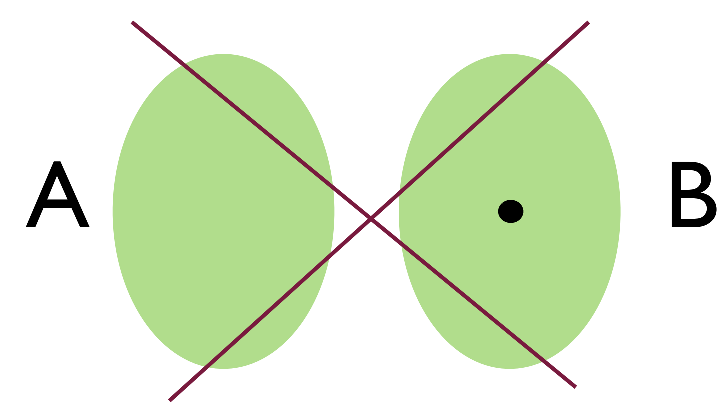
Def. A function $f: A \longrightarrow B$ is injective iff

for all $a, b \in A$, if $f(a) = f(b)$ then $a = b$.



Def. A function $f: A \longrightarrow B$ is surjective iff

for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.



Def. A function $f: A \longrightarrow B$ is bijective iff

f is injective and surjective.

Simple characterisations

Lemma I1: A function $f: A \longrightarrow B$ is injective iff

for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

at most one incoming arrow
injection

Lemma S1: A function $f: A \longrightarrow B$ is surjective iff

$|f^{-1}(\{b\})| \geq 1$ for all $b \in B$ iff
$f(A) = B$.

at least one incoming arrow
surjection

Lemma B1: A function $f: A \longrightarrow B$ is bijective iff

$|f^{-1}(\{b\})| = 1$ for all $b \in B$ iff
f is both injective and surjective.

exactly one incoming arrow
bijection

Some properties

Lemma 12: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$$f(x) \in f(A') \text{ iff } x \in A'.$$

if holds always!

Prop. 13: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

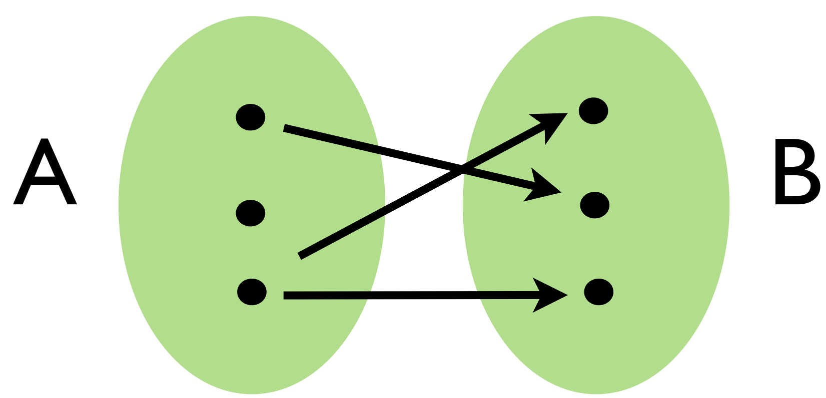
$$f^{-1}(f(A')) = A'.$$

Prop. S2: Let $f: A \longrightarrow B$ be surjective and let $B' \subseteq B$. Then

$$f(f^{-1}(B')) = B'.$$

Inverse function

Let $f: A \longrightarrow B$ be a bijection



well defined only if f is bijective!

Def. The inverse function $f^{-1}: B \longrightarrow A$ is defined as

$$f^{-1}(b) = a \text{ iff } f(a) = b, \quad b \in B.$$

Lemma B2: The inverse function f^{-1} of a bijection f is bijective.

Function composition

Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

"after"

$g \circ f : A \longrightarrow B \longrightarrow C$

Def. The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by

$$g \circ f\,(a) = g(f(a)), \quad \text{for } a \in A.$$

Lemma I4: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be injective. Then $g \circ f$ is injective.

Lemma S3: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be surjective. Then $g \circ f$ is surjective.

Corollary B2: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be bijective. Then so is $g \circ f$.

A characterization of bijections

Theorem B3: A function $f: A \longrightarrow B$ is bijective iff

there exists a function $g: B \longrightarrow A$ with

$g \circ f = id_A$ and $f \circ g = id_B$.

$id_A: A \longrightarrow A$,

$id_A(a) = a$, for all $a \in A$