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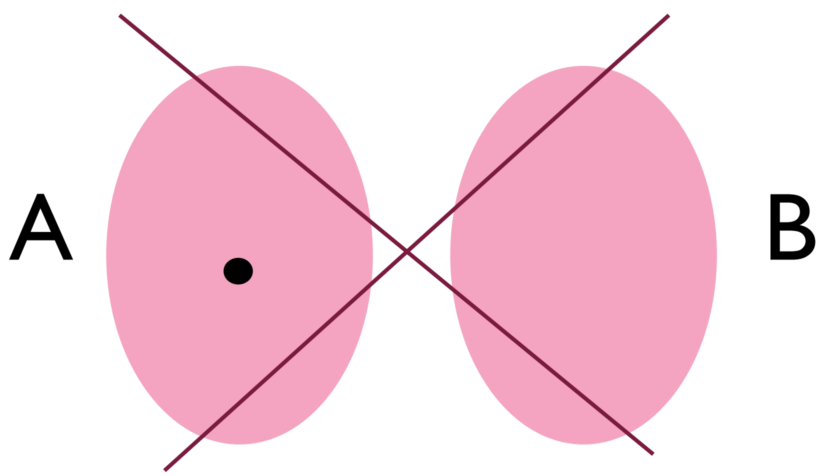

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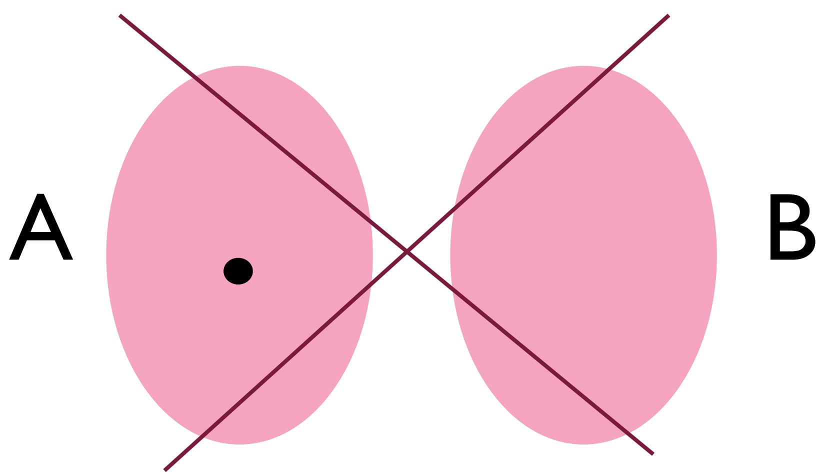

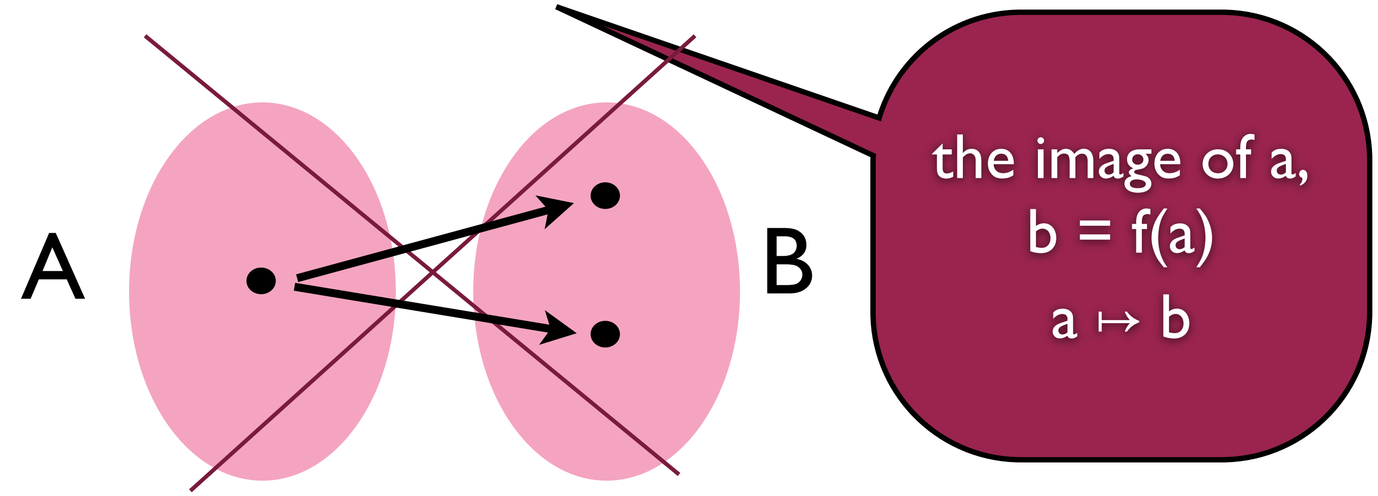

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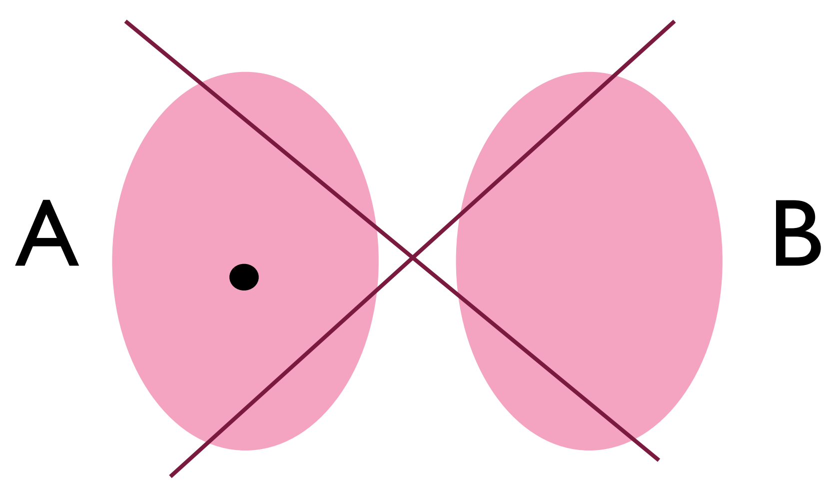

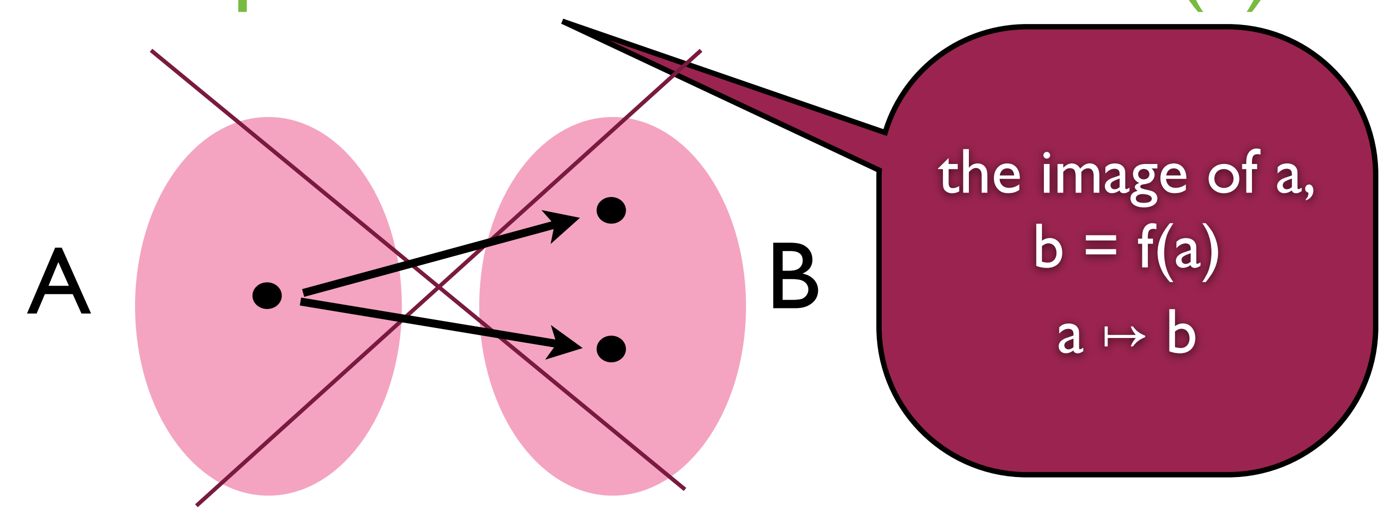

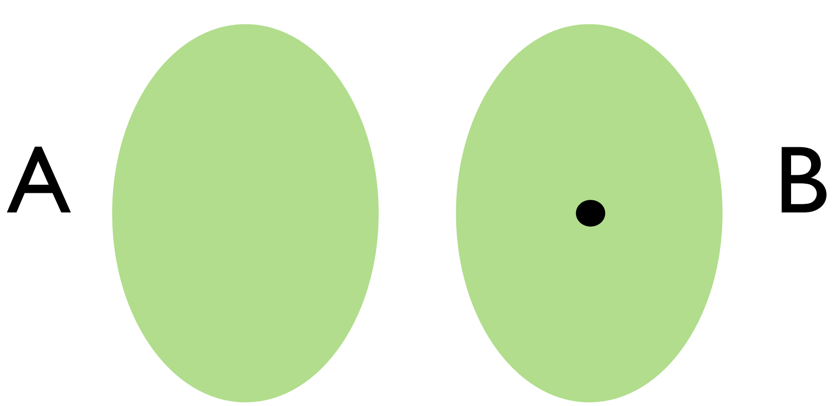

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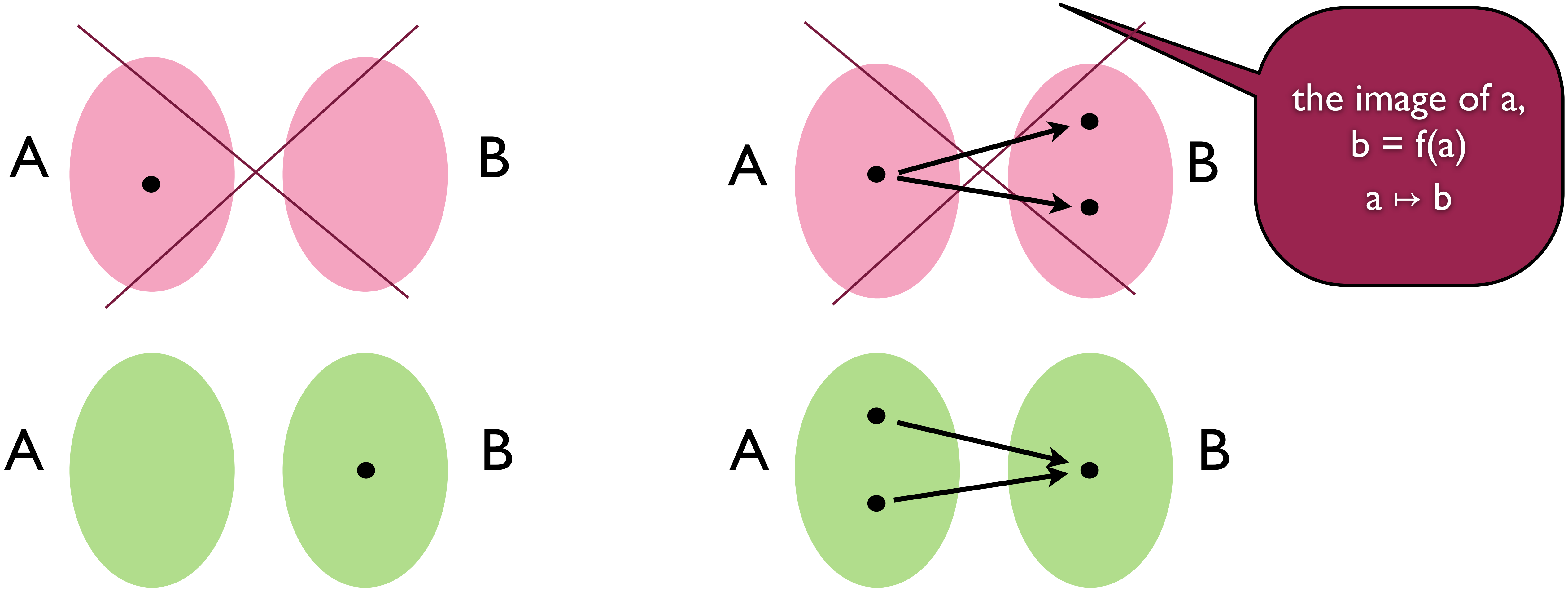

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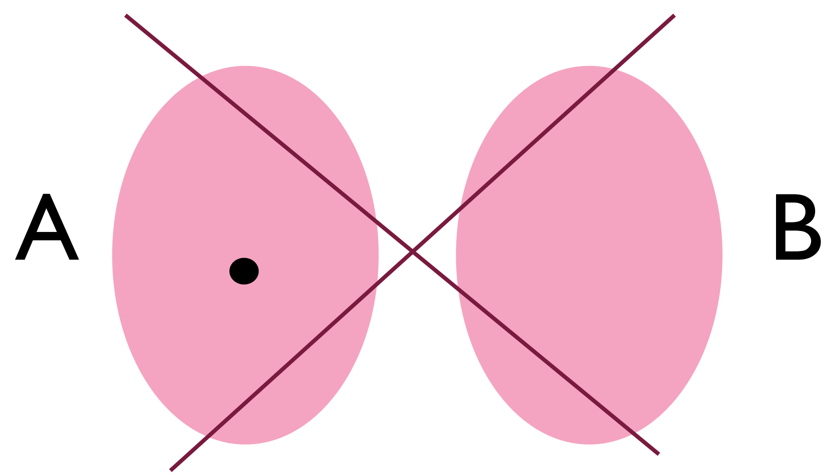

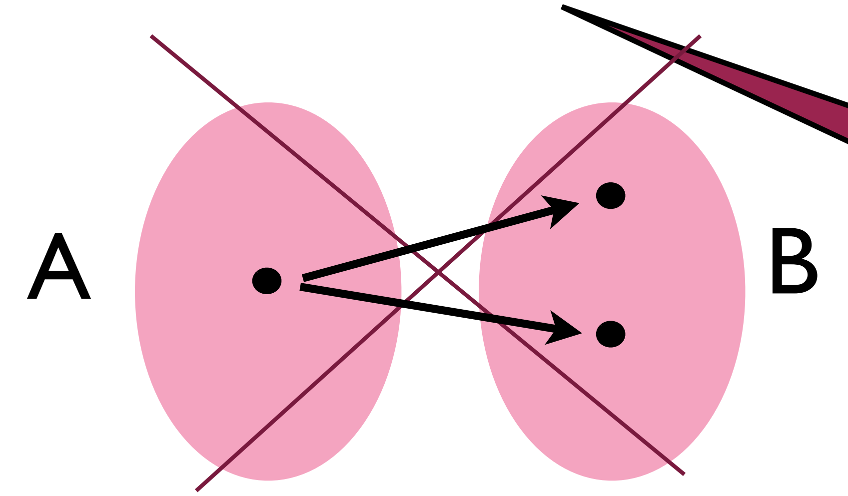

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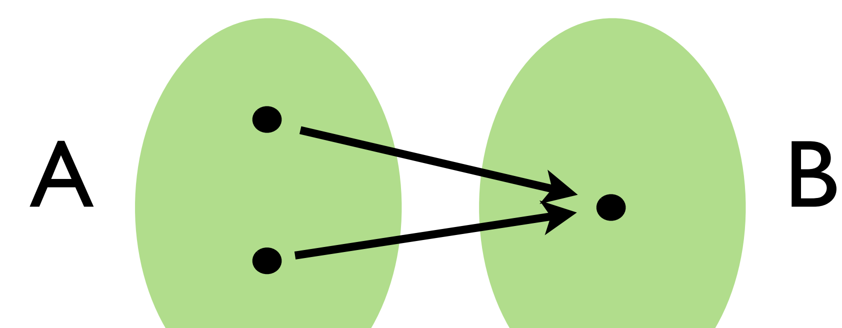

$\{(a, f(a)) \mid a \in A\}$ is the graph of the function f

Functions, mappings

When $f\colon A \longrightarrow B$ then $\mathrm{dom}\, f = A$ and $\mathrm{cod}\, f = B$

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When f: $A \longrightarrow B$ then dom f = A and cod f = B

domain of F
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Equality of functions

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So f extends to a function $f\colon \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$, the image-function.

Inverse image

Let $f\colon A \longrightarrow B$ and $B' \subseteq B$.

The inverse image (Urbild) of B' is the set

$$f^{-1}(B') = \{a \mid f(a) \in B'\} \subseteq A.$$

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Again the inverse image induces a function $f^{-1}: \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$, the inverse-image-function.

Lemma F1: Let f: $A \longrightarrow B$, $A' \subseteq A$, and $B' \subseteq B$. Then

$$A' \subseteq f^{-1}(f(A')) \quad \text{and} \quad f(f^{-1}(B')) \subseteq B'$$

(in general no more than this holds)

Recall…

Def. If A and B are sets, a function f from A to B, notation

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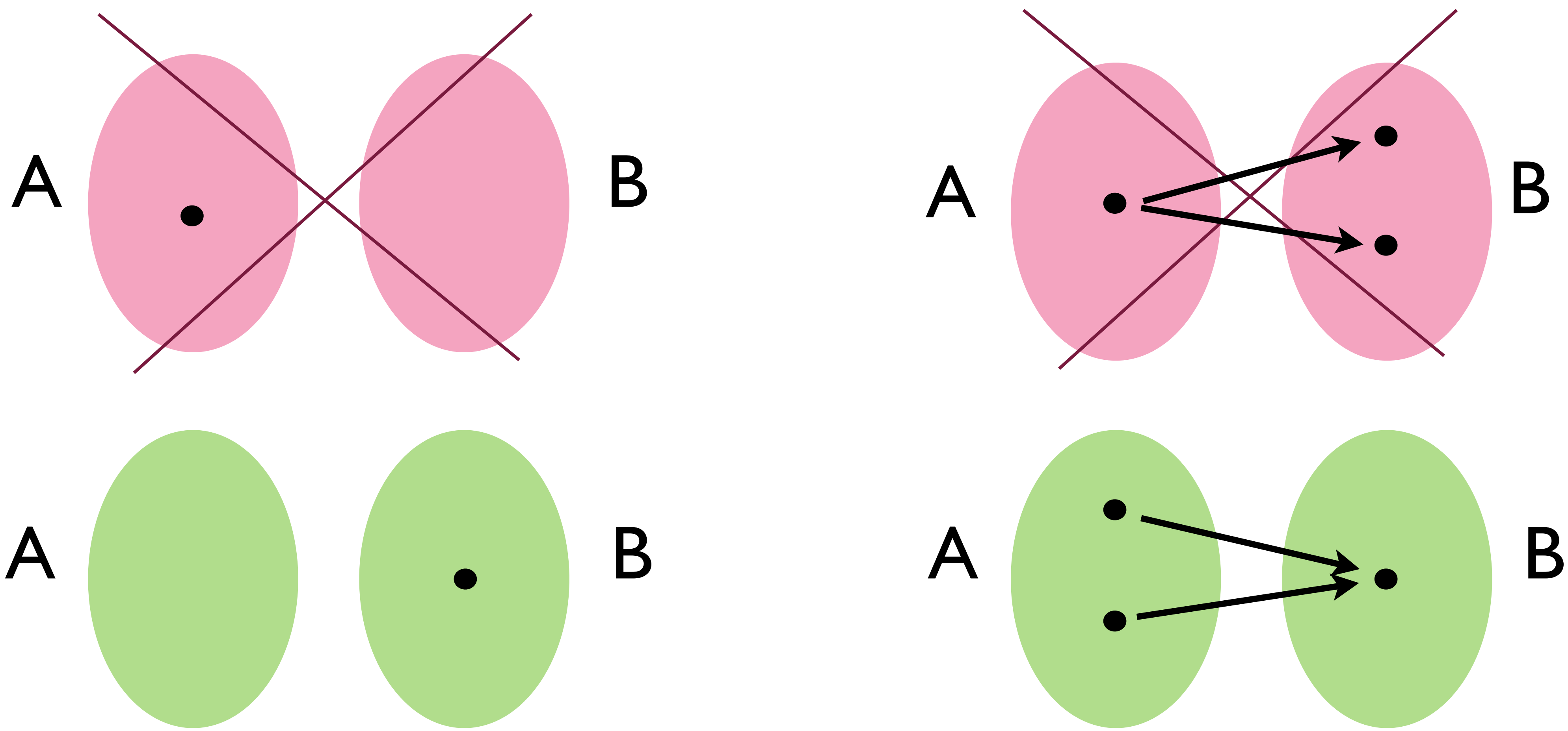

Special functions

The number of ingoing arrows for a function can be 0, 1, or more.
Based on this, we distinguish some special functions.

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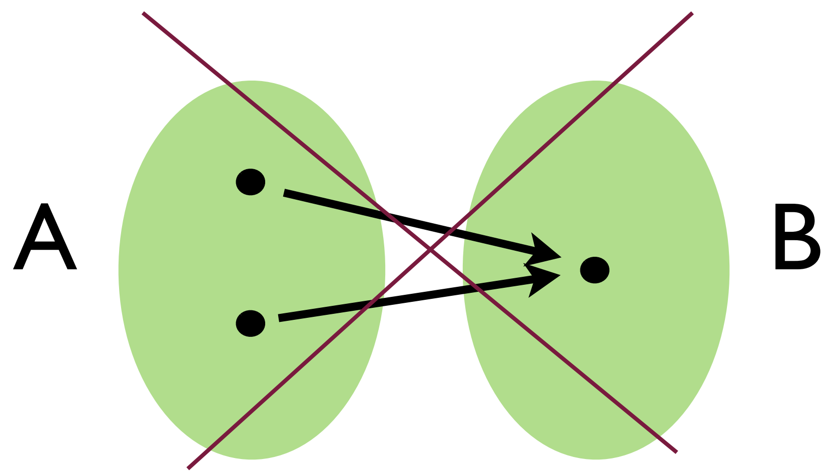

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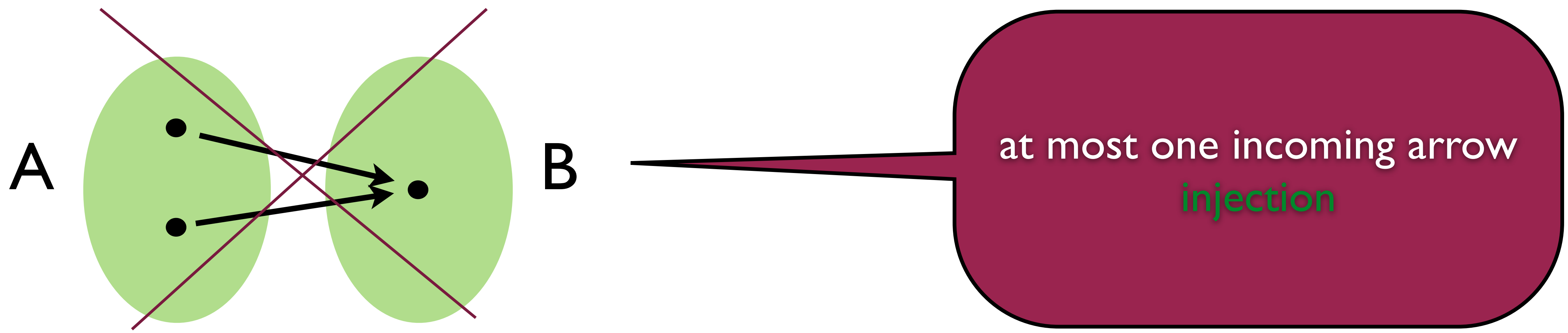

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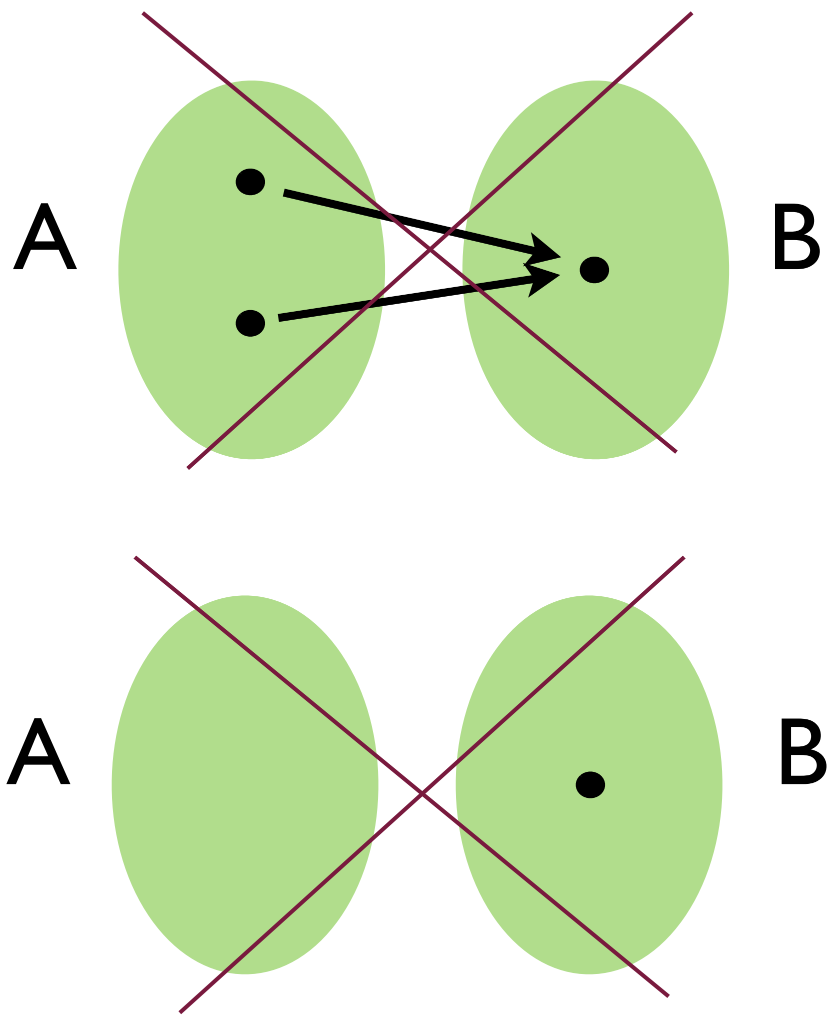

at most one incoming arrow
injection

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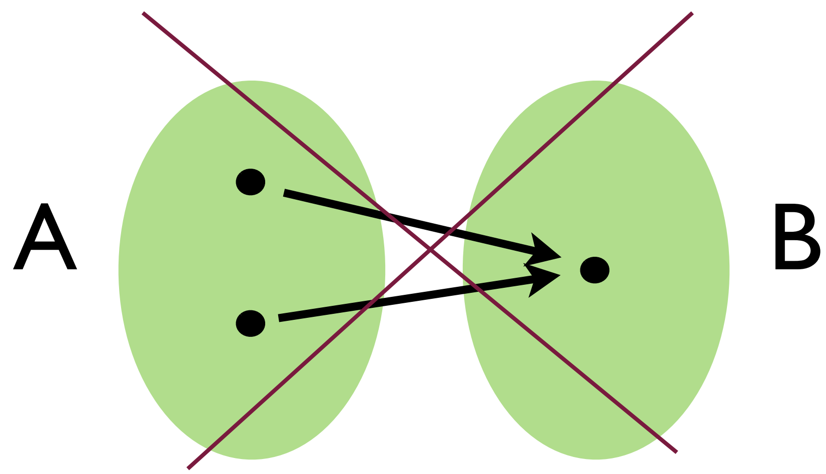

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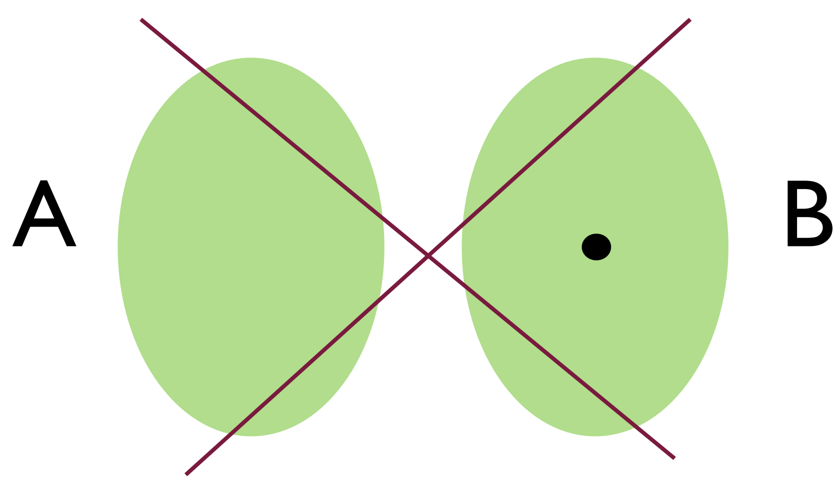

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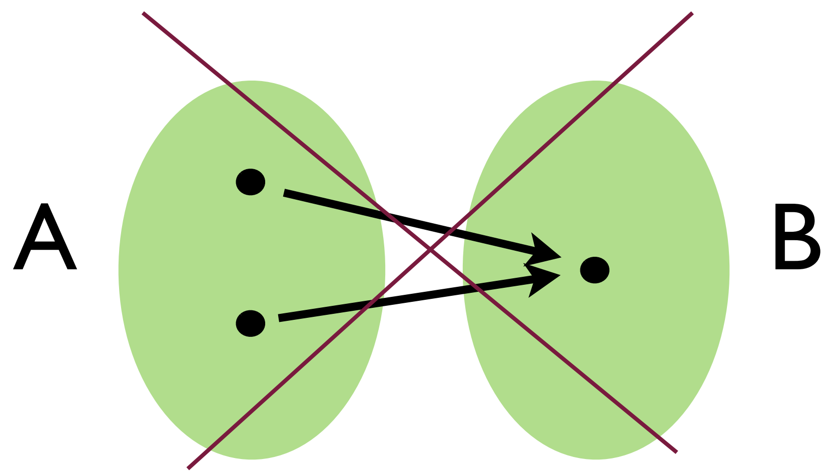

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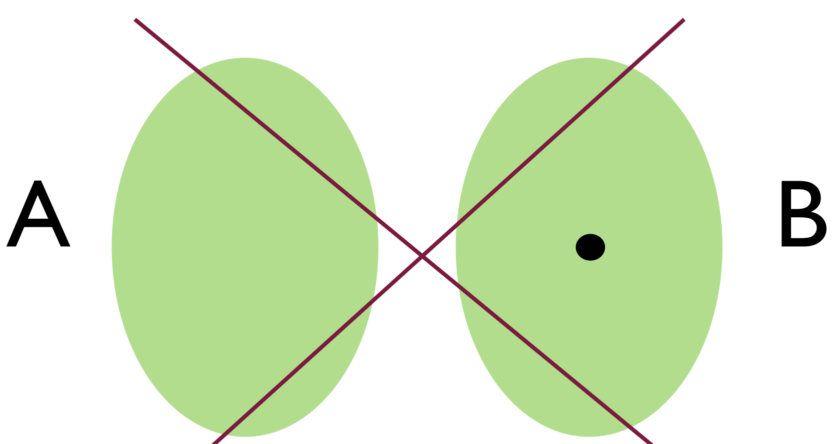

at least one incoming arrow
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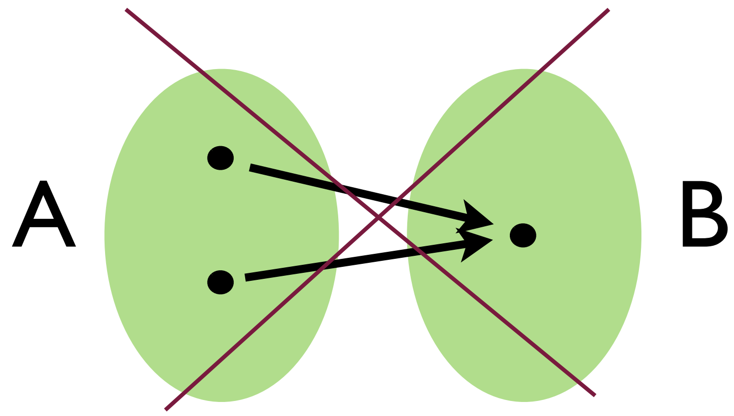
exactly one incoming arrow (injection + surjection) bijection

Special functions

Special functions

Def. A function $f\colon A \longrightarrow B$ is injective iff

for all $a, b \in A$, if $f(a) = f(b)$ then $a = b$.



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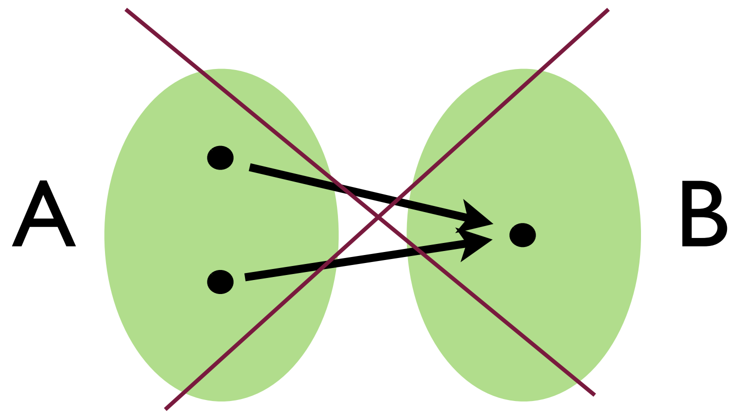
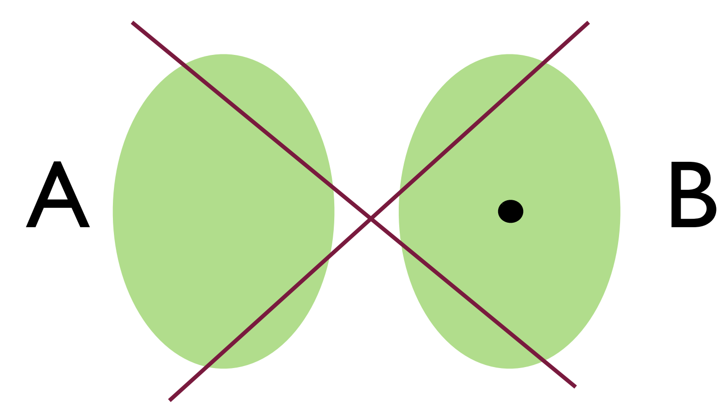


Def. A function $f: A \longrightarrow B$ is surjective iff

for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.



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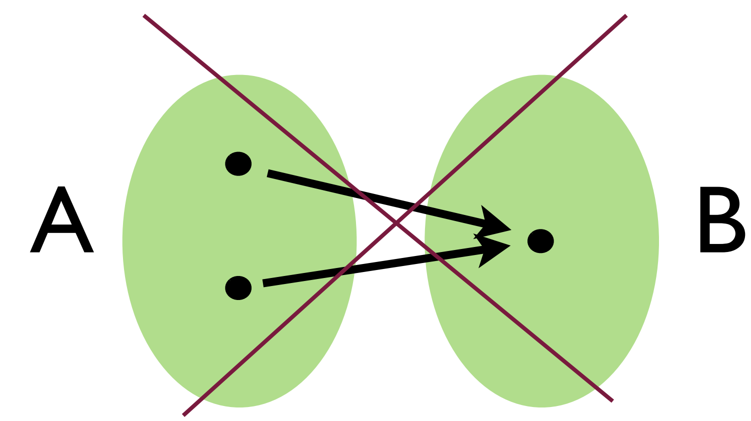

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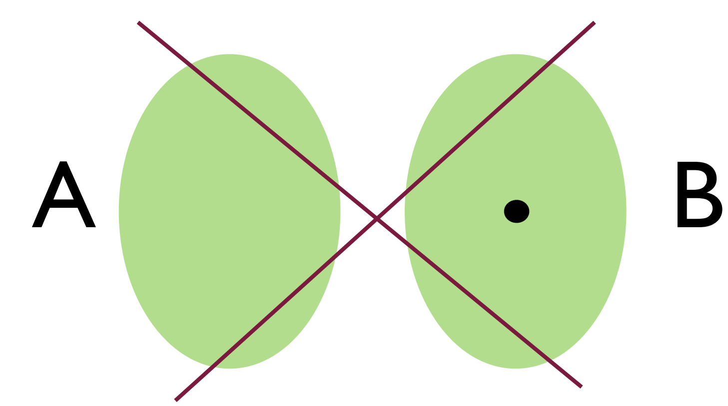

Def. A function $f: A \longrightarrow B$ is bijective iff

f is injective and surjective.

Simple characterisations

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Lemma 11: A function $f : A \longrightarrow B$ is injective iff

for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

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at most one incoming arrow
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Simple characterisations

Lemma I1: A function $f: A \longrightarrow B$ is injective iff

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at most one incoming arrow
injection

Lemma S1: A function $f: A \longrightarrow B$ is surjective iff

$|f^{-1}(\{b\})| \geq 1$ for all $b \in B$ iff

$f(A) = B$.

Simple characterisations

Lemma I1: A function $f: A \longrightarrow B$ is injective iff

$$\text{for all } b \in B,\ |f^{-1}(\{b\})| \leq 1.$$

at most one incoming arrow
injection

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at least one incoming arrow
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at least one incoming arrow

surjection

Lemma B1: A function $f : A \longrightarrow B$ is bijective iff

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Simple characterisations

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at most one incoming arrow
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exactly one incoming arrow
bijection

Some properties

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Lemma 12: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$$f(x) \in f(A') \text{ iff } x \in A'.$$

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if holds always!

Prop. 13: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f^{-1}(f(A')) = A'$.

Some properties

Lemma I2: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f(x) \in f(A')$ iff $x \in A'$.

if holds always!

Prop. I3: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

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Prop. S2: Let $f: A \longrightarrow B$ be surjective and let $B' \subseteq B$. Then

$f(f^{-1}(B')) = B'$.

Inverse function

Let f:A $\longrightarrow$ B be a bijection

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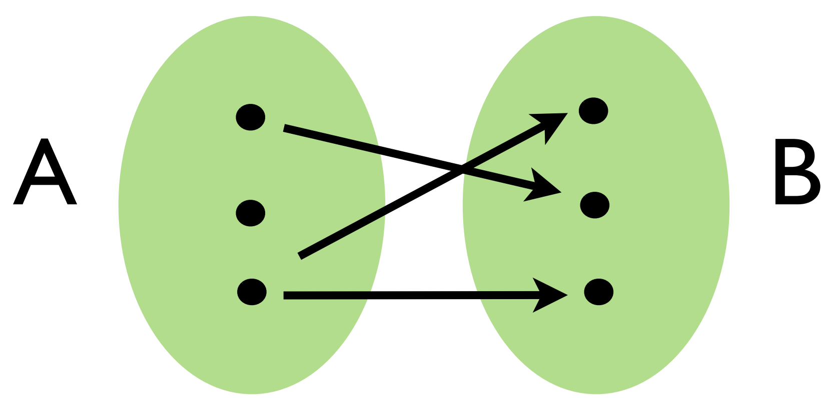

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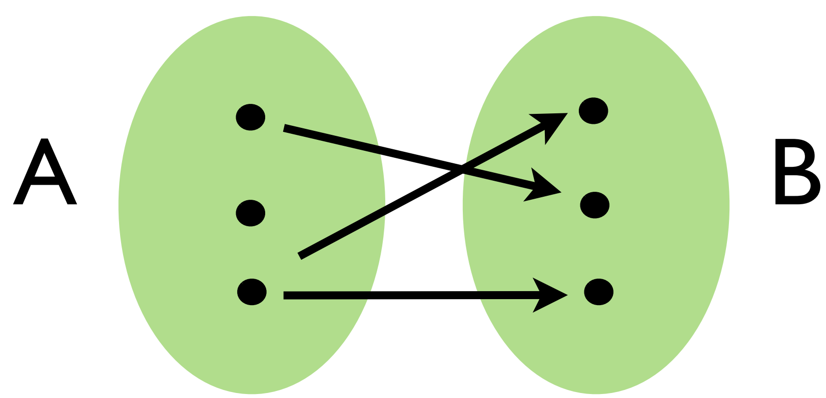

Def. The inverse function $f^{-1}: B \longrightarrow A$ is defined as

$$f^{-1}(b) = a \quad \text{iff} \quad f(a) = b, \quad b \in B.$$

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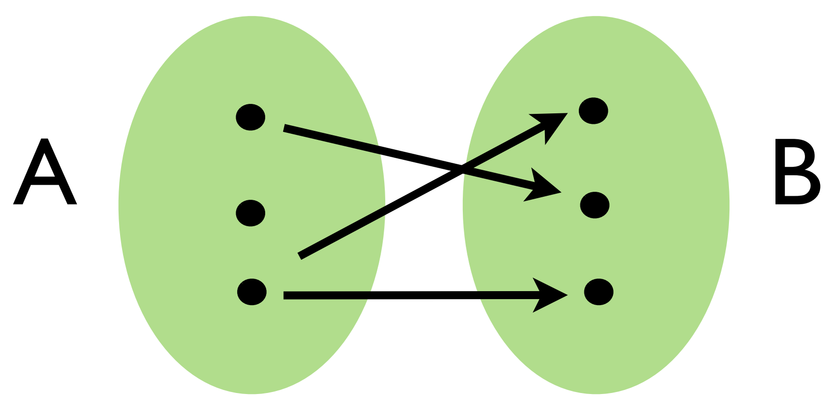

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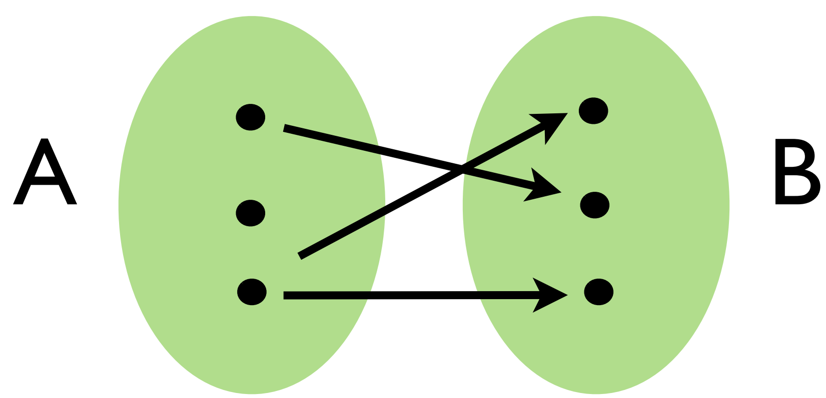

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Def. The inverse function $f^{-1}{:}\,B \longrightarrow A$ is defined as

$$f^{-1}(b) = a \ \text{ iff } \ f(a) = b, \quad b \in B.$$

Lemma B2: The inverse function f^{-1} of a bijection f is bijective.

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

Function composition

Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

Def. The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by

$$g \circ f\,(a) = g(f(a)), \text{ for } a \in A.$$

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

"after"

$g \circ f : A \longrightarrow B \longrightarrow C$

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Lemma 14: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be injective. Then $g \circ f$ is injective.

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

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Def. The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by

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Lemma I4: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be injective. Then

$g \circ f$ is injective.

Lemma S3: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be surjective. Then

$g \circ f$ is surjective.

Function composition

Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$

"after"
$g \circ f : A \longrightarrow B \longrightarrow C$

Def. The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by

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Lemma I4: Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be injective. Then

$g \circ f$ is injective.

Lemma S3: Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be surjective. Then

$g \circ f$ is surjective.

Corollary B2: Let $f\colon A \longrightarrow B$ and $g\colon B \longrightarrow C$ be bijective. Then so is $g \circ f$.

A characterization of bijections

A characterization of bijections

Theorem B3: A function $f: A \longrightarrow B$ is bijective iff there exists a function $g: B \longrightarrow A$ with $g \circ f = id_A$ and $f \circ g = id_B$.

A characterization of bijections

Theorem B3: A function $f: A \longrightarrow B$ is bijective iff

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$id_A: A \longrightarrow A$,

$id_A(a) = a$, for all $a \in A$