Derivations / Reasoning

Limitations of proofs by calculation

Proofs by calculation are formal and well-structured, but often undirected and not particularly intuitive.

Example

$$
\begin{aligned}
P \wedge (P \vee Q) &\overset{val}{=} (P \vee F) \wedge (P \vee Q) \\
&\overset{val}{=} P \vee (F \wedge Q) \\
&\overset{val}{=} P \vee F \\
&\overset{val}{=} P
\end{aligned}
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Conclusions

$P \wedge (P \vee Q) \overset{val}{=} P$ $\quad$ $P \wedge (P \vee Q) \Leftrightarrow P \overset{val}{=} T$

Limitations of proofs by calculation

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P \land (P \lor Q) &\overset{val}{=} (P \lor F) \land (P \lor Q) \\
&\overset{val}{=} P \lor (F \land Q) \\
&\overset{val}{=} P \lor F \\
&\overset{val}{=} P
\end{aligned}
$$

we can prove this more intuitively by reasoning

Conclusions

$P \land (P \lor Q) \overset{val}{=} P$ $\quad$ $P \land (P \lor Q) \Leftrightarrow P \overset{val}{=} T$

An example of a mathematical proof

Theorem

If x^2 is even, then x is even ($x \in \mathbb{Z}$).

Proof

Let $x \in \mathbb{Z}$ be such that x^2 is even.

We need to prove that x is even too.

Assume that x is odd, towards a contradiction.

If x is odd than $x = 2y+1$ for some $y \in \mathbb{Z}$.

Then $x^2 = (2y+1)^2 = 4y^2 + 4y + 1 = 2(2y^2 + 2y) + 1$
and $2y^2 + 2y \in \mathbb{Z}$.

So, x^2 is odd too, and we have a contradiction.

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(sub)goal

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generating hypothesis

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An example of a mathematical proof

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(sub)goal

generating hypothesis

pure hypothesis

conclusion

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(sub)goal

generating hypothesis

pure hypothesis

conclusion

Thanks to Bas Luttik

Exposing logical structure

Theorem If x^2 is even, then x is even ($x \in \mathbb{Z}$).

Proof

Let $x \in \mathbb{Z}$

Assume x^2 is even.

Assume that x is odd.

Then $x = 2y+1$ for some $y \in \mathbb{Z}$.

Then $x^2 = (2y+1)^2 = 4y^2 + 4y + 1 = 2(2y^2 + 2y) + 1$ and $2y^2 + 2y \in \mathbb{Z}$.

So, x^2 is odd

a contradiction.

So, x is even

(sub)goal

generating hypothesis

pure hypothesis

conclusion

Thanks to Bas Luttik

Single inference rule

Q is a correct conclusion from n premises $P_1, .., P_n$

iff

$(P_1 \wedge P_2 \wedge \ldots \wedge P_n) \overset{val}{\models} Q$

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Single inference rule

Q is a correct conclusion from n premises $P_1, .., P_n$
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Note that $T \vDash Q$ means that $Q \overset{val}{=} T$

Single inference rule

Q is a correct conclusion from n premises $P_1, .., P_n$
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If n=0, then $P_1 \wedge P_2 \wedge \ldots \wedge P_n \overset{val}{=} T$

Note that $T \models Q$ means that $Q \overset{val}{=} T$

Q holds unconditionally

Derivation

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a formal system
based on the single
inference rule
for proofs that closely
follow our
intuitive reasoning

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Two types of inference rules:

elimination rules

introduction rules

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(particularly useful) instances of the single inference rule

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(particularly useful) instances of the single inference rule

and one new special rule!

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a formal system based on the single inference rule for proofs that closely follow our intuitive reasoning

Two types of inference rules:

elimination rules

for drawing conclusions out of premises

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Derivation

a formal system based on the single inference rule for proofs that closely follow our intuitive reasoning

Q is a correct conclusion from n premises $P_1, .., P_n$
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Two types of inference rules:

(particularly useful) instances of the single inference rule

elimination rules

for drawing conclusions out of premises

introduction rules

for simplifying goals

and one new special rule!

Conjunction elimination

How do we use a conjunction in a proof?

Conjunction elimination

How do we use a conjunction in a proof?

$P \wedge Q \overset{val}{\vDash} P$

$P \wedge Q \overset{val}{\vDash} Q$

Conjunction elimination

How do we use a conjunction in a proof?

$P \wedge Q \overset{val}{\models} P$

$P \wedge Q \overset{val}{\models} Q$

	‖ ‖
(k)	$P \wedge Q$
	‖ ‖
	{$\wedge$-elim on (k)}
(m)	P

(k < m)

Conjunction elimination

How do we use a conjunction in a proof?

$P \wedge Q \overset{val}{\vDash} P$

$P \wedge Q \overset{val}{\vDash} Q$

|| ||

(k) $P \wedge Q$

|| ||

{∧-elim on (k)}
(m) P

(k < m)

|| ||

(k) $P \wedge Q$

|| ||

{∧-elim on (k)}
(m) Q

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Conjunction elimination

How do we use a conjunction in a proof?

$P \wedge Q \overset{\text{val}}{\models} P$

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$\wedge$-elimination

|| ||

(k) $P \wedge Q$

|| ||

{$\wedge$-elim on (k)}

(m) P

(k < m)

|| ||

(k) $P \wedge Q$

|| ||

{$\wedge$-elim on (k)}

(m) Q

(k < m)

Implication elimination

How do we use an implication in a proof?

Implication elimination

How do we use an implication in a proof?

$P \Rightarrow Q \overset{val}{\models}$???

$(P \Rightarrow Q) \wedge P \overset{val}{\models} Q$

Implication elimination

How do we use an implication in a proof?

$P \Rightarrow Q \overset{val}{\models}$???

$(P \Rightarrow Q) \wedge P \overset{val}{\models} Q$

|| ||

(k) $P \Rightarrow Q$

|| ||

(l) P

|| ||

{$\Rightarrow$-elim on (k) and (l)}

(m) Q

$(k < m, l < m)$

Implication elimination

How do we use an implication in a proof?

$P \Rightarrow Q \overset{val}{\models}$???

$(P \Rightarrow Q) \wedge P \overset{val}{\models} Q$

⇒-elimination

|| ||

(k) $P \Rightarrow Q$

|| ||

(l) P

|| ||

{⇒-elim on (k) and (l)}

(m) Q

$(k < m, l < m)$

Conjunction introduction

How do we prove a conjunction?

Conjunction introduction

How do we prove a conjunction?

$P \wedge Q \overset{val}{\models} P \wedge Q$

Conjunction introduction

How do we prove a conjunction?

$P \wedge Q \overset{val}{\models} P \wedge Q$

…

(k) P

…

(l) Q

…

{∧-intro on (k) and (l)}

(m) $P \wedge Q$

(k < m, l < m)

Conjunction introduction

How do we prove a conjunction?

$P \wedge Q \overset{val}{\models} P \wedge Q$

$\wedge$-introduction

…

(k) P

…

(l) Q

…

{$\wedge$-intro on (k) and (l)}

(m) $P \wedge Q$

$(k < m, l < m)$

Implication introduction

How do we prove an implication?

Implication introduction

How do we prove an implication?

...

{Assume}

(k) P

...

(l-1) Q

{⇒-intro on (k) and (l-1)}

(l) P⇒Q

Implication introduction

How do we prove an implication?

⇒-introduction

…

{Assume}

(k) P

…

(l-1) Q

{⇒-intro on (k) and (l-1)}

(l) $P \Rightarrow Q$

Implication introduction

How do we prove an implication?

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{⇒-intro on (k) and (l-1)}

(l) $P \Rightarrow Q$

flag shows the validity of a hypothesis

Implication introduction

How do we prove an implication?

truly new and necessary for reasoning with hypothesis

$\Rightarrow$-introduction

. . .

{Assume}

(k) P

. . .

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truly new
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flag shows the validity of a hypothesis

time for an example!

Negation introduction

How do we prove a negation?

Negation introduction

How do we prove a negation?

$\neg P \overset{val}{=} P \Rightarrow F$

Negation introduction

How do we prove a negation?

$\neg P \overset{val}{=} P \Rightarrow F$

...
{Assume}
(k) P

...

(l-1) F
{¬-intro on (k) and (l-1)}
(l) ¬P

Negation introduction

How do we prove a negation?

$\neg P \overset{val}{=} P \Rightarrow F$

¬-introduction

```
           ...
           {Assume}
(k)      | P
         |
         |  ...
         |
(l-1)    | F
           {¬-intro on (k) and (l-1)}
(l)      ¬P
```

Negation introduction

How do we prove a negation?

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⇒-intro

¬-introduction

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Negation elimination

How do we use a negation in a proof?

Negation elimination

How do we use a negation in a proof?

$P \wedge \neg P \overset{val}{\vDash} F$

Negation elimination

How do we use a negation in a proof?

$P \wedge \neg P \overset{val}{\models} F$

||| |||

(k) P

||| |||

(l) ¬P

||| |||

{¬-elim on (k) and (l)}

(m) F

(k < m, l < m)

Negation elimination

How do we use a negation in a proof?

$P \wedge \neg P \overset{val}{\models} F$

¬-elimination

|| ||

(k) P

|| ||

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|| ||

(k) P

|| ||

(l) ¬P

|| ||

{¬-elim on (k) and (l)}

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(k < m, l < m)

time for an example!

F introduction

How do we prove F?

F introduction

How do we prove F?

$P \wedge \neg P \overset{val}{\models} F$

F introduction

How do we prove F?

$P \wedge \neg P \overset{val}{\models} F$

...

(k) P

...

(l) ¬P

...

{F-intro on (k) and (l)}

(m) F

$(k < m, l < m)$

F introduction

How do we prove F?

$P \wedge \neg P \overset{val}{\models} F$

F-introduction

. . .

(k) P

. . .

(l) ¬P

. . .

{F-intro on (k) and (l)}

(m) F

(k < m, l < m)

F introduction

How do we prove F?

$P \wedge \neg P \overset{val}{\models} F$

F-introduction

...

(k) P

...

(l) ¬P

...

{F-intro on (k) and (l)}

(m) F

the same as ¬-elim
only intended bottom-up

$(k < m, l < m)$

F elimination

How do we use F in a proof?

F elimination

How do we use F in a proof?

it's very useful!

$F \overset{val}{\models} P$

F elimination

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$F \overset{val}{\models} P$

||||

(k) F

||||

{F-elim on (k)}
(m) P

(k < m)

F elimination

How do we use F in a proof?

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$F \overset{val}{\models} P$

F-elimination

	‖ ‖	
(k)	F	
	‖ ‖	
	{F-elim on (k)}	
(m)	P	

(k < m)

Double negation introduction

How do we prove ¬¬?

Double negation introduction

How do we prove ¬¬?

$P \overset{val}{\models} \neg\neg P$

Double negation introduction

How do we prove ¬¬?

P $\overset{val}{\models}$ ¬¬P

...

(k) P

...

{¬¬-intro on (k)}

(m) ¬¬P

(k < m)

Double negation introduction

How do we prove ¬¬?

¬¬-introduction

$P \overset{val}{\models} \neg\neg P$

...

(k) P

...

{¬¬-intro on (k)}

(m) ¬¬P

(k < m)

Double negation elimination

How do we use ¬¬ in a proof?

Double negation elimination

How do we use ¬¬ in a proof?

$\neg\neg P \overset{val}{\models} P$

Double negation elimination

How do we use ¬¬ in a proof?

$\neg\neg P \overset{val}{\models} P$

|| ||

(k) ¬¬P

|| ||

{¬¬-elim on (k)}
(m) P

(k < m)

Double negation elimination

How do we use ¬¬ in a proof?

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¬¬-elimination

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(k) ¬¬P

|| ||

{¬¬-elim on (k)}
(m) P

(k < m)

Proof by contradiction

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If x^2 is even, then x is even ($x \in \mathbb{Z}$).

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Assume x^2 is even.

Assume that x is odd.

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Then $x^2 = (2y+1)^2 = 4y^2 + 4y + 1 = 2(2y^2 + 2y) + 1$ and $2y^2 + 2y \in \mathbb{Z}$.

So, x^2 is odd

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So, x is even

(sub)goal

generating hypothesis

pure hypothesis

conclusion

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conclusion

Thanks to Bas Luttik

Proof by contradiction

How do we prove P by a contradiction?

Proof by contradiction

How do we prove P by a contradiction?

{Assume}
(k) ¬P

…

(l-1) F
{¬-intro on (k) and (l-1)}
(l) ¬¬P
{¬¬-elim on (l)}
(l+1) P

(k < m)

Proof by contradiction

How do we prove P by a contradiction?

proof by contradiction

	{Assume}
(k)	$\neg P$
	...
(l-1)	F
	{$\neg$-intro on (k) and (l-1)}
(l)	$\neg\neg P$
	{$\neg\neg$-elim on (l)}
(l+1)	P

(k < m)

Proof by contradiction

How do we prove P by a contradiction?

$\neg P \Rightarrow F \overset{val}{\models} \neg\neg P \overset{val}{\models} P$

proof by contradiction

{Assume}

(k) $\neg P$

…

(l-1) F

{¬-intro on (k) and (l-1)}

(l) $\neg\neg P$

{¬¬-elim on (l)}

(l+1) P

(k < m)

Proof by contradiction

How do we prove P by a contradiction?

$\neg P \Rightarrow F \overset{val}{\models} \neg\neg P \overset{val}{\models} P$

¬-intro

proof by contradiction

{Assume}

(k) ¬P

...

(l-1) F

{¬-intro on (k) and (l-1)}

(l) ¬¬P

{¬¬-elim on (l)}

(l+1) P

(k < m)

Proof by contradiction

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¬-intro

¬¬-elim

proof by contradiction

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...

(l-1) F

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(l) $\neg\neg P$

{¬¬-elim on (l)}

(l+1) P

$(k < m)$

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$\neg P \Rightarrow F \overset{val}{\models} \neg\neg P \overset{val}{\models} P$

¬-intro

¬¬-elim

proof by contradiction

{Assume}

(k) ¬P

...

(l-1) F

{¬-intro on (k) and (l-1)}

(l) ¬¬P

{¬¬-elim on (l)}

(l+1) P

(k < m)

time for an example!

Disjunction introduction

How do we prove a disjunction?

Disjunction introduction

How do we prove a disjunction?

$\neg P \Rightarrow Q \overset{val}{\models} P \vee Q$

$\neg Q \Rightarrow P \overset{val}{\models} P \vee Q$

Disjunction introduction

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$\neg P \Rightarrow Q \overset{val}{\models} P \vee Q$

$\neg Q \Rightarrow P \overset{val}{\models} P \vee Q$

...
{Assume}
(k) ¬P
...
(l-1) Q
{∨-intro on (k) and (l-1)}
(l) P∨Q

Disjunction introduction

How do we prove a disjunction?

$\neg P \Rightarrow Q \overset{val}{\models} P \vee Q$

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∨-introduction

...

{Assume}

(k) ¬P

...

(l-1) Q

{∨-intro on (k) and (l-1)}

(l) P∨Q

Disjunction introduction

How do we prove a disjunction?

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∨-introduction

...

{Assume}

(k) ¬P

...

(l-1) Q

{∨-intro on (k) and (l-1)}

(l) P∨Q

⇒-intro

Disjunction introduction

How do we prove a disjunction?

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∨-introduction

...
{Assume}
(k) ¬Q

...

(l-1) P
{∨-intro on (k) and (l-1)}
(l) P∨Q

⇒-intro

Disjunction elimination

How do we use a disjunction in a proof?

Disjunction elimination

How do we use a disjunction in a proof?

$P \vee Q \overset{val}{\models} \neg P \Rightarrow Q$

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Disjunction elimination

How do we use a disjunction in a proof?

$P \vee Q \overset{val}{\models} \neg P \Rightarrow Q$

$P \vee Q \overset{val}{\models} \neg Q \Rightarrow P$

|| ||

(k) $P \vee Q$

|| ||

{∨-elim on (k)}

(m) $\neg P \Rightarrow Q$

$(k < m)$

Disjunction elimination

How do we use a disjunction in a proof?

$P \vee Q \overset{val}{\models} \neg P \Rightarrow Q$

$P \vee Q \overset{val}{\models} \neg Q \Rightarrow P$

∨-elimination

||||

(k) $P \vee Q$

||||

{∨-elim on (k)}

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Disjunction elimination

How do we use a disjunction in a proof?

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∨-elimination

||||

(k) $P \vee Q$

||||

{∨-elim on (k)}

(m) $\neg Q \Rightarrow P$

(k < m)

Proof by case distinction

How do we prove R by a case distinction?

Proof by case distinction

How do we prove R by a case distinction?

|| ||

(k) $P \vee Q$

|| ||

(l) $P \Rightarrow R$

|| ||

(m) $Q \Rightarrow R$

|| ||

{case-dist on (k), (l), (m)}
(n) R

(k < n, l< n, m<n)

Proof by case distinction

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proof by
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Proof by case distinction

How do we prove R by a case distinction?

$(P \vee Q) \wedge (P \Rightarrow R) \wedge (Q \Rightarrow R) \overset{val}{\models} R$

proof by
case distinction

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(k) $P \vee Q$

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(m) $Q \Rightarrow R$

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Bi-implication introduction

How do we prove a bi-implication?

$(P \Rightarrow Q) \wedge (Q \Rightarrow P) \overset{val}{\models} P \Leftrightarrow Q$

⇔-introduction

Bi-implication introduction

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$(P \Rightarrow Q) \wedge (Q \Rightarrow P) \stackrel{val}{\models} P \Leftrightarrow Q$

⇔-introduction

…

(k) $P \Rightarrow Q$

…

(l) $Q \Rightarrow P$

…

{⇔-intro on (k) and (l)}

(m) $P \Leftrightarrow Q$

$(k < m, l < m)$

Bi-implication introduction

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Bi-implication elimination

How do we use a bi-implication in a proof?

Bi-implication elimination

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$P \Leftrightarrow Q \overset{val}{\models} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

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{⇔-elim on (k)}

(m) $P \Rightarrow Q$

(k < m)

Bi-implication elimination

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Bi-implication elimination

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$P \Leftrightarrow Q \overset{val}{\vDash} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$

∧-elim

⇔-elimination

	⋮ ⋮
(k)	$P \Leftrightarrow Q$
	⋮ ⋮
	{⇔-elim on (k)}
(m)	$P \Rightarrow Q$

(k < m)

	⋮ ⋮
(k)	$P \Leftrightarrow Q$
	⋮ ⋮
	{⇔-elim on (k)}
(m)	$Q \Rightarrow P$

(k < m)

Derivations / Reasoning with quantifiers

Proving a universal quantification

To prove $\forall x[x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$

Proving a universal quantification

To prove

$\forall x[x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$

Proof

Let $x \in \mathbb{Z}$ be arbitrary and assume that $x \geq 2$.

Then, for this particular x, it holds that

$$x^2 - 2x = x(x-2) \geq 0 \quad \text{(Why?)}$$

Conclusion: $\forall x[x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$.

∀ introduction

How do we prove a universal quantification?

∀ introduction

How do we prove a universal quantification?

	...
	{Assume}
(k)	**var** x; P(x)
	...
(l-1)	Q(x)
	{∀-intro on (k) and (l-1)}
(l)	∀x[P(x) : Q(x)]

∀ introduction

How do we prove a universal quantification?

∀-introduction

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flag shows the validity of a hypothesis

∀ introduction

How do we prove a universal quantification?

similar to ⇒-intro

with generating hypothesis

∀-introduction

...

{Assume}

(k) **var** x; P(x)

...

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{∀-intro on (k) and (l-1)}

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flag shows the validity of a hypothesis

Using a universal quantification

We know $\forall x[x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$

Using a universal quantification

We know $\forall x[x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$

Whenever we encounter an $a \in \mathbb{Z}$ such that $a \geq 2$,

we can conclude that $a^2 - 2a \geq 0$.

For example, $(52387^2 - 2 \cdot 52387) \geq 0$
since $52387 \in \mathbb{Z}$ and $52387 \geq 2$.

∀ elimination

How do we use a universal quantification in a proof?

∀ elimination

How do we use a universal quantification in a proof?

similar to implication but we need a witness

∀ elimination

How do we use a universal quantification in a proof?

similar to implication but we need a witness

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(k) $\forall x[P(x) : Q(x)]$

|| ||

(l) $P(a)$

|| ||

{∀-elim on (k) and (l)}

(m) $Q(a)$

$(k < m, l < m)$

∀ elimination

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∀-elimination

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a is an object (variable, number,..) which is "known" in line (l)

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the same "a" from line (l)

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a is an object (variable, number,..) which is "known" in line (l)

the same "a" from line (l)

time for an example!

$(k < m, l < m)$

∃ introduction

How do we prove an existential quantification?

∃ introduction

How do we prove an existential quantification?

$\neg\ \forall x[P(x):\neg Q(x)] \overset{val}{\models} \exists x\ [P(x) : Q(x)]$

∃ introduction

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$\neg \forall x[P(x) : \neg Q(x)] \overset{val}{\models} \exists x\, [P(x) : Q(x)]$

…

{Assume}

(k) $\forall x[P(x) : \neg Q(x)]$

…

(l-1) F

{∃-intro on (k) and (l-1)}

(l) $\exists x\, [P(x) : Q(x)]$

∃ introduction

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(l-1) F

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and ¬-intro

∃ elimination

How do we use an existential quantification in a proof?

∃ elimination

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$\exists x\ [P(x) : Q(x)] \overset{val}{\models}$
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|| ||

{∃-elim on (k) and (l)}

(m) F

and ¬-elimination

(k < m, l < m)

∃ elimination

How do we use an existential quantification in a proof?

∃-elimination

|| ||

(k) ∃x [P(x) : Q(x)]

|| ||

(l) ∀x[P(x): ¬Q(x)]

|| ||

{∃-elim on (k) and (l)}

(m) F

(k < m, l < m)

∃x [P(x) : Q(x)] $\overset{val}{\models}$ ¬ ∀x[P(x):¬Q(x)]

and ¬-elimination

∃ elimination

How do we use an existential quantification in a proof?

$\exists x\,[P(x) : Q(x)] \overset{val}{\models} \neg\, \forall x[P(x){:}\neg Q(x)]$

∃-elimination

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(l) $\forall x[P(x){:}\ \neg Q(x)]$

|| ||

{∃-elim on (k) and (l)}

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and ¬-elimination

time for an example!

Proofs with ∃-introduction and ∃-elimination are unnecessarily long and cumbersome…

Proofs with ∃-introduction and ∃-elimination are unnecessarily long and cumbersome…

There are alternatives!

Proving an existential quantification

To prove $\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$

Proving an existential quantification

To prove

$\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$

Proof

It suffices to find a witness, i.e., an $x \in \mathbb{Z}$ satisfying

$$x^3 - 2x - 8 \geq 0.$$

$x = 3$ is a witness, since $3 \in \mathbb{Z}$ and $3^3 - 2 \cdot 3 - 8 = 13 \geq 0$

Conclusion: $\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$.

Proving an existential quantification

To prove

$\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$

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$x = 3$ is a witness, since $3 \in \mathbb{Z}$ and $3^3 - 2 \cdot 3 - 8 = 13 \geq 0$

Conclusion: $\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$.

also x = 5 is a witness…

Alternative ∃ introduction

How do we prove an existential quantification?

Alternative ∃ introduction

How do we prove an existential quantification?

by finding
a witness

Alternative ∃ introduction

How do we prove an existential quantification?

by finding a witness

	...
(k)	P(a)
	...
(l)	Q(a)
	...
	{∃*-intro on (k) and (l)}
(m)	∃x [P(x) : Q(x)]

(k < m, l < m)

Alternative ∃ introduction

How do we prove an existential quantification?

by finding a witness

∃∗-introduction

$\ldots$

(k) P(a)

$\ldots$

(l) Q(a)

$\ldots$

{∃∗-intro on (k) and (l)}

(m) ∃x [P(x) : Q(x)]

(k < m, l < m)

Alternative ∃ introduction

How do we prove an existential quantification?

by finding
a witness

∃∗-introduction

…

(k) $P(a)$

…

(l) $Q(a)$

…

{∃∗-intro on (k) and (l)}

(m) $\exists x\ [P(x) : Q(x)]$

strategy: wait until a witness object appears

$(k < m, l < m)$

Alternative ∃ introduction

How do we prove an existential quantification?

by finding a witness

∃∗-introduction

...

(k) P(a)

...

(l) Q(a)

...

{∃∗-intro on (k) and (l)}

(m) ∃x [P(x) : Q(x)]

strategy: wait until a witness object appears

does not always work

(k < m, l < m)

Using an existential quantification

We know

$\exists x[x \in \mathbb{R} : a - x < 0 < b - x]$

Using an existential quantification

We know $\exists x[x \in \mathbb{R} : a - x < 0 < b - x]$

We can declare an $x \in \mathbb{Z}$ (a witness) such that

$$a - x < 0 < b - x$$

and use it further in the proof. For example:

From $a - x < 0$, we get $a < x$.

From $b - x > 0$, we get $x < b$.

Hence, $a < b$.

Alternative ∃ elimination

How do we use an existential quantification in a proof?

Alternative ∃ elimination

How do we use an existential quantification in a proof?

we pick a witness

Alternative ∃ elimination

How do we use an existential quantification in a proof?

we pick a witness

|| ||

(k) ∃x [P(x) : Q(x)]

|| ||

{∃*-elim on (k)}

(m) Pick x with P(x) and Q(x)

(k < m)

Alternative ∃ elimination

How do we use an existential quantification in a proof?

we pick a witness

∃∗-elimination

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∃*-elimination

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x must be new!

(k < m)

Alternative ∃ elimination

How do we use an existential quantification in a proof?

we pick a witness

∃∗-elimination

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(k < m)