Predicate logic

Limitations of propositional logic

Propositional logic only allows us to reason about completed statements about things, not about the things themselves.

Example

Some chicken cannot fly
All chicken are birds

Some birds cannot fly

this reasoning can not be expressed in propositional logic

Example

Every player except the winner looses a match

Unary predicate (example)

Consider the statement $2m>3$.

Whether this statement is true or false depends on the value of m (and on the domain of values).

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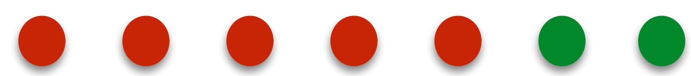

on $\mathbb{Z}$

… -3 -2 -1 0 1 2 3 …

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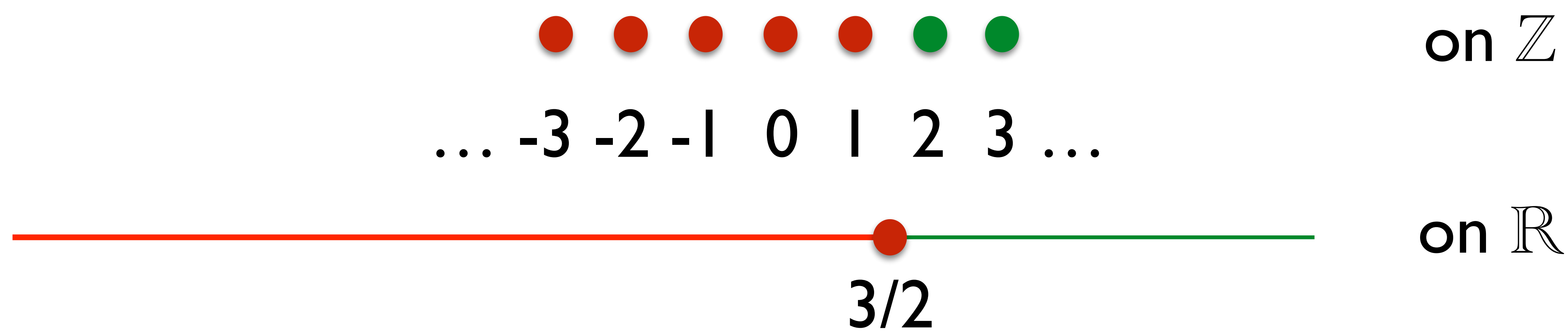

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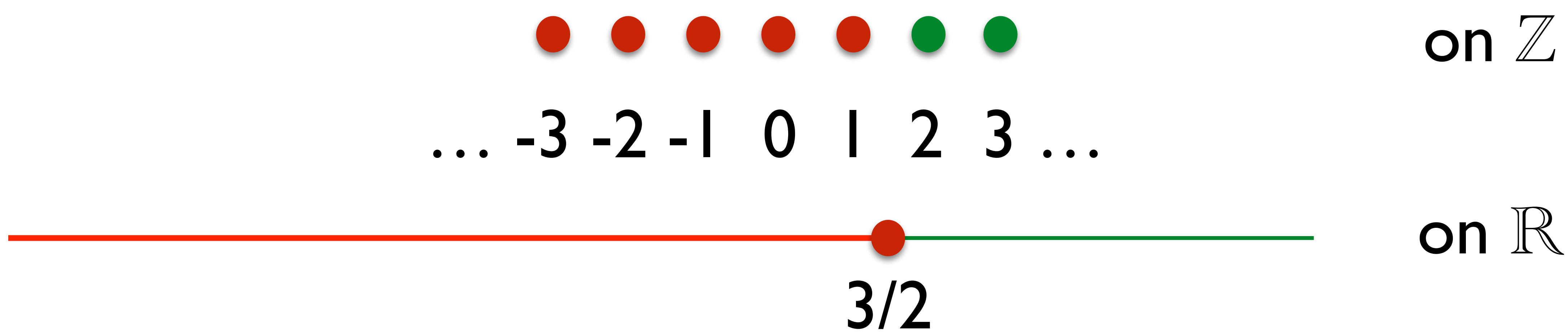

Note: $2m > 3 \overset{\text{val}}{=} m > 3/2$ on $\mathbb{Z}$ and $\mathbb{R}$

$2m > 3 \overset{\text{val}}{=} m \geq 2$ on $\mathbb{Z}$ but not on $\mathbb{R}$

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a unary relation

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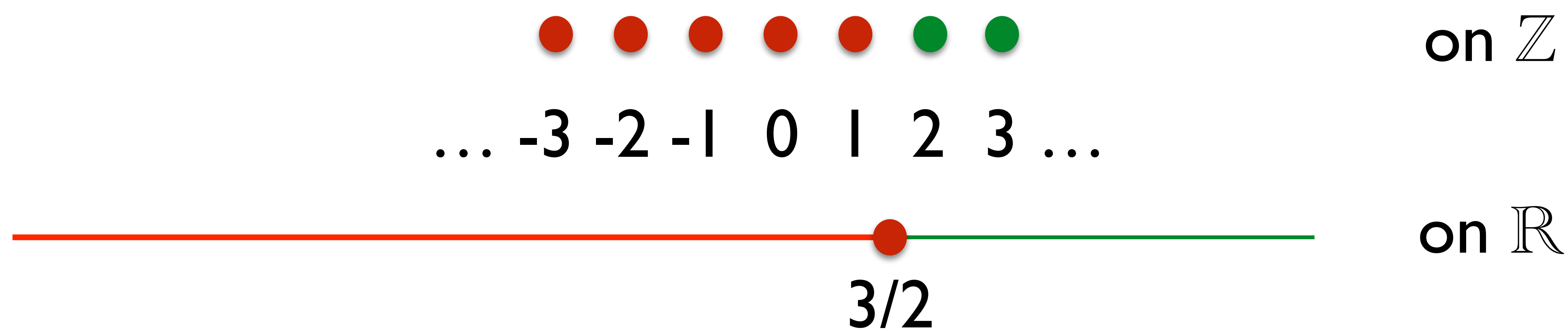

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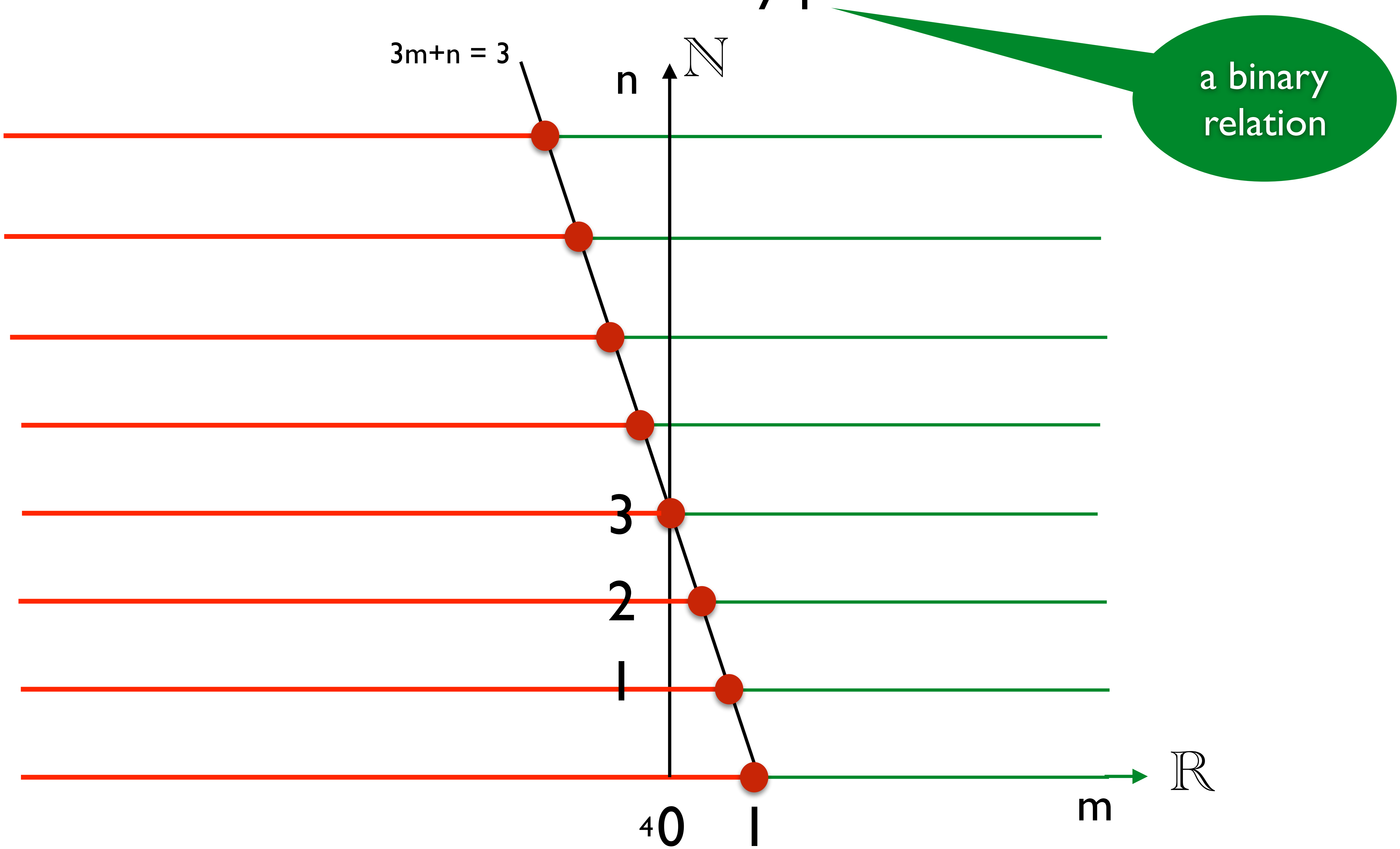
Binary predicate (example)

The statement $3m+n > 3$ is a binary predicate on $\mathbb{R} \times \mathbb{N}$.



Predicates

In general, an n-ary predicate is an n-ary relation.

If it is on a domain D, then it's a relation $P(x_1, .., x_n) \subseteq D^n$ or equivalently a function $P: D^n \rightarrow \{0,1\}$.

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We can turn a predicate, into a proposition in three ways:

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for $m=2$

$2 \cdot 2 > 3$

is a true proposition

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other standard (!) notation:

$\exists m\ (m \in \mathbb{R} \wedge \forall n\ (n \in \mathbb{N} \Rightarrow 3m+n>3))$

Additional Notation Rules

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We write $\forall_x\ [P]$ for $\forall_x\ [T : P]$

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but only for the same quantifier!

Quantification - task

Let P be the set of all tennis players.
Let $w \in P$ be the winner.

For $p, q \in P$, write $p \neq q$ for "p and q are different players" .

Let M be the set of all matches.
For $p \in P$ and $m \in M$, write $L(p,m)$ for
"player p loses match m".

Write the following sentence as a formula with predicates and quantifiers:

Every player except the winner loses a match.

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Thanks to Bas Luttik

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Equivalences with quantifiers

Renaming bound variables

Bound variables

$$\forall_x[P{:}Q] \stackrel{val}{=} \forall_y[P[y/x]{:}Q[y/x]]$$

$$\exists_x[P{:}Q] \stackrel{val}{=} \exists_y[P[y/x]{:}Q[y/x]]$$

if y does not occur in
P or Q (not even in ∀y, ∃y)

Domain splitting

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$$\forall_x[P \vee Q : R] \stackrel{val}{=} \forall_x[P : R] \wedge \forall_x[Q : R]$$

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Examples:

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Examples:

$$\forall_x[x \leqslant 1 \vee x \geqslant 5 \colon x^2 - 6x + 5 \geqslant 0]$$
$$\stackrel{val}{=} \forall_x[x \leqslant 1 \colon x^2 - 6x + 5 \geqslant 0] \wedge \forall_x[x \geqslant 5 \colon x^2 - 6x + 5 \geqslant 0]$$

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$$\begin{aligned} &\exists_k[0 \leqslant k \leqslant n : k^2 \leqslant 10] \\ \overset{val}{=}\ &\exists_k[0 \leqslant k \leqslant n-1 \vee k = n : k^2 \leqslant 10] \\ \overset{val}{=}\ &\exists_k[0 \leqslant k \leqslant n-1 : k^2 \leqslant 10] \vee \exists_k[k = n : k^2 \leqslant 10] \end{aligned}$$

Equivalences with quantifiers

One-element domain

$$\forall_x[x = n : Q] \stackrel{val}{=} Q[n/x]$$

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Empty domain

$$\forall_x[F : Q] \overset{val}{=} T$$

$$\exists_x[F : Q] \overset{val}{=} F$$

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"All Marsians are green"

Domain weakening

Intuition: The following are equivalent

$$\forall_x[x \in D : A(x)] \quad \text{and} \quad \forall_x[x \in D \Rightarrow A(x)]$$

$$\exists_x[x \in D : A(x)] \quad \text{and} \quad \exists_x[x \in D \wedge A(x)]$$

The same can be done to parts of the domain

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$$\forall_x[P \wedge Q : R] \stackrel{val}{=} \forall_x[P : Q \Rightarrow R]$$
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$$P \wedge Q \stackrel{val}{\models} P$$

De Morgan with quantifiers

De Morgan

$$\neg\forall_x[P:Q] \stackrel{val}{=} \exists_x[P:\neg Q]$$
$$\neg\exists_x[P:Q] \stackrel{val}{=} \forall_x[P:\neg Q]$$

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Hence: $\neg\forall = \exists\neg$ and $\neg\exists = \forall\neg$

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Hence: $\neg\forall = \exists\neg$ and $\neg\exists = \forall\neg$

It holds further that:

$$\neg\forall_x\neg = \exists_x\neg\neg = \exists_x$$
$$\neg\exists_x\neg = \forall_x\neg\neg = \forall_x$$

Substitution

meta rule

Simple

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P] \overset{val}{=} \psi[\xi/P]}$$

EVERY occurrence of P is substituted!

Sequential

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{=} \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{=} \psi[\xi/P, \eta/Q]}$$

Substitution

holds also for quantified formulas!

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The rule of Leibniz

meta rule

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$$\frac{\phi \stackrel{val}{=} \psi}{C[\phi] \stackrel{val}{=} C[\psi]}$$

formula that has ϕ as a sub formula

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The rule of Leibniz

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meta rule

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Other equivalences with quantifiers

Exchange trick

$$\forall_x[P{:}Q] \overset{val}{=} \forall_x[\neg Q{:}\neg P]$$

$$\exists_x[P{:}Q] \overset{val}{=} \exists_x[Q{:}P]$$

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No wonder as

$$\forall_x[P{:}Q] \stackrel{val}{=} \forall_x[P \Rightarrow Q]$$

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Term splitting

$$\forall_x[P{:}Q \wedge R] \stackrel{val}{=} \forall_x[P{:}Q] \wedge \forall_x[P{:}R]$$

$$\exists_x[P{:}Q \vee R] \stackrel{val}{=} \exists_x[P{:}Q] \vee \exists_x[P{:}R]$$

Other equivalences with quantifiers

Monotonicity of quantifiers

$$\forall_x[P{:}Q \Rightarrow R] \Rightarrow (\forall_x[P{:}Q] \Rightarrow \forall_x[P{:}R]) \overset{val}{=} T$$

$$\forall_x[P{:}Q \Rightarrow R] \Rightarrow (\exists_x[P{:}Q] \Rightarrow \exists_x[P{:}R]) \overset{val}{=} T$$

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tautologies

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tautologies

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Lemma W5: If $Q \stackrel{val}{\models} R$ then $\forall_x[P{:}Q] \stackrel{val}{\models} \forall_x[P{:}R]$.