

The structure of natural numbers

is helpful for proving
properties

$$\forall n[n \in \mathbb{N} : P(n)]$$

The structure of natural numbers

On natural numbers we can define a notion of a **successor**, a mapping

$$s: \mathbb{N} \rightarrow \mathbb{N}$$

by $s(n) = n+1$

The successor mapping imposes a structure on the set that enables us to **count**:

- 1) there is a **starting** natural number 0
- 2) for every natural number n , there is a **next** natural number $s(n) = n+1$.

(Some) Peano Axioms

Important properties

(1) Different natural numbers have different successors:

$$\forall n, m [n, m \in \mathbb{N} : s(m) = s(n) \Rightarrow m = n]$$

stated positively

s is injective!

(2) 0 is not a successor: $\forall n [n \in \mathbb{N} : \neg (s(n) = 0)]$

(3) All natural numbers except 0 are successors:

$$\forall n [n \in \mathbb{N} \wedge \neg(n = 0) : \exists m [m \in \mathbb{N} : n = s(m)]]$$

There is more to it - induction

Imagine an infinite sequence of dominos



If we know that

1. D_0 falls
2. The dominos are close enough together so that if D_i falls, then D_{i+1} falls (for all $i \in \mathbb{N}$)

Then we can conclude that every domino D_n ($n \in \mathbb{N}$) falls!



induction

Induction

P - unary predicate
over \mathbb{N}

$$P(0) \wedge \forall i [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n [n \in \mathbb{N} : P(n)]$$

\forall elim
with 0

\Rightarrow elim



$$P(0)$$
$$P(0) \Rightarrow P(1)$$

$$P(1)$$
$$P(1) \Rightarrow P(2)$$

$$P(2)$$
$$P(2) \Rightarrow P(3)$$

...

Variant of the Peano Axiom:

Let $K \subseteq \mathbb{N}$ have the property that

(a) $0 \in K$ and

(b) for all $n \in \mathbb{N}$, $n \in K \Rightarrow (n+1) \in K$.

Then $K = \mathbb{N}$.

Induction

$$P(0) \wedge \forall i [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n [n \in \mathbb{N} : P(n)]$$

P - unary predicate
over \mathbb{N}

...

(m) P(0)
{Assume}

(k) **var** i; i \in \mathbb{N}

(k+1) P(i)
...

(l-1) P(i+1)
{ \Rightarrow -intro on (k+1) and (l-1)}

(l) P(i) \Rightarrow P(i+1)
{ \forall -intro on (k) and (l)}

(l+1) $\forall i[i \in \mathbb{N} : P(i) \Rightarrow P(i+1)]$
{induction on (m) and (l+1)}

(l+2) $\forall n[n \in \mathbb{N} : P(n)]$

Basis

induction
hypothesis

Induction step

Inductive definitions

Inductive proof: truth is passed on

Inductive definition: construction is passed on

well defined by induction

Example

The sequence of real numbers $(a_i \mid i \in \mathbb{N})$ is defined inductively by

$$a_0 = 2$$

$$a_{i+1} = 2a_i - 1$$

a_0	a_1	a_2	a_3	a_4	...
2	3	5	9	17	...

proof by induction

Conjecture

For all $n \in \mathbb{N}$ it holds that

$$a_n = 2^{n+1}$$

Strong induction

P - unary predicate
over \mathbb{N}

$$\forall k [k \in \mathbb{N} : \forall j [j \in \mathbb{N} \wedge j < k : P(j)] \Rightarrow P(k)] \Rightarrow \forall n [n \in \mathbb{N} : P(n)]$$

\forall elim with $k=1$

\Rightarrow elim,
 \wedge intro



$$\begin{aligned} &P(0) \\ &P(0) \Rightarrow P(1) \\ &P(0) \wedge P(1) \\ &P(0) \wedge P(1) \Rightarrow P(2) \\ &P(0) \wedge P(1) \wedge P(2) \\ &P(0) \wedge P(1) \wedge P(2) \Rightarrow P(3) \\ &\dots \end{aligned}$$

Definition of
 $(a_i \mid i \in \mathbb{N})$
with strong
induction

a_n is defined via
 a_0, \dots, a_{n-1}

Cardinality

Cardinals

Def.

Two sets A and B have the same cardinality (are equinumerous) if there is a bijection $f:A \rightarrow B$.
Notation $A \sim B$, or $|A| = |B|$.

Prop.

The relation \sim is an equivalence relation on sets.

Def.

A set A has at most as large cardinality as a set B if there is an injection $f:A \rightarrow B$.
Notation $|A| \leq |B|$.

Def.

A set A has at least as large cardinality as a set B if B is empty or there is a surjection $f:A \rightarrow B$.
Notation $|A| \geq |B|$.

Def.

A set A has smaller cardinality than a set B if there is an injection $f:A \rightarrow B$ and there is no surjection $f:A \rightarrow B$. Notation $|A| < |B|$.

$$|A| = [A]_{\sim}$$

cardinal
numbers are
 \sim equivalence
classes

Theorem (Cantor)

If $|A| \leq |B|$
and
 $|B| \leq |A|$,
then
 $|A| = |B|$.

Operations on cardinals

Def.

Let A and B be two disjoint sets. Then $|A| + |B| = |A \cup B|$.

Def.

Let A and B be two sets. Then $|A| \cdot |B| = |A \times B|$.

Def.

Let A and B be two sets. Then $|A|^{|B|} = |A^B|$ where A^B is the set of all functions from B to A , i.e. $A^B = \{f \mid f: B \rightarrow A\}$.

Prop.

Let A be a set. Then $|\mathcal{P}(A)| = 2^{|A|}$.

$$|A| = [A]_{\sim}$$

cardinal numbers are \sim equivalence classes

Note: $2 = |\{0, 1\}|$

Finite sets, finite cardinals

We write \mathbb{N}_k for the set $\{0, 1, \dots, k-1\}$. Then $\mathbb{N}_0 = \emptyset$.

We will also write k for $|\mathbb{N}_k|$.

Def.

A set A is finite if and only if $|A| = |\mathbb{N}_k|$, for some $k \in \mathbb{N}$. We write then $|A| = k$.

Hence

A set A is finite if and only if there is a natural number $k \in \mathbb{N}$ and a bijection $f: A \rightarrow \mathbb{N}_k$.

$$|A| = [A]_{\sim}$$

cardinal numbers are \sim equivalence classes

if and only if A has k elements, for some $k \in \mathbb{N}$

E.g. If $|A| = k$ and $|B| = m$ for some $k, m \in \mathbb{N}$ then $|A \times B| = k \cdot m$

The operations on cardinals when restricted to finite cardinals coincide with the operations on natural numbers!
This justifies the notation.

Infinite, countable and uncountable sets

Time for a video!

Hilbert's
infinite hotel :-)

Infinite, countable and uncountable sets

We write \aleph_0 for the cardinality of natural numbers.
Hence $\aleph_0 = |\mathbb{N}|$.

Def.

A set A is countable iff $|A| = \aleph_0$.

Prop.

\mathbb{N} is countable.
 \mathbb{Z} is countable.
 \mathbb{Q} is countable.

Def.

A set is infinite iff $|A| \geq \aleph_0$.

Def.

A set is uncountable iff $|A| > \aleph_0$.

Prop.

\mathbb{R} is uncountable.

$$|A| = [A]_{\sim}$$

cardinal numbers are \sim equivalence classes

Hence, every countable set is infinite

We write c for $|\mathbb{R}|$

Cardinals are unbounded

Theorem (Cantor)

For every set A we have $|A| < |\mathcal{P}(A)|$.

Hence, for every cardinal there is a larger one.

$$|A| = [A]_{\sim}$$

cardinal
numbers are
 \sim equivalence
classes