# The structure of natural numbers

is helpful for proving properties

$\forall n[n \in \mathbb{N}: P(n)]$

# The structure of natural numbers

On natural numbers we can define a notion of a successor, a mapping

s: $\mathbb{N} \to \mathbb{N}$

by s(n) = n+1

The successor mapping imposes a structure on the set that enables us to count:

1) there is a starting natural number 0
2) for every natural number n, there is a next natural number s(n) = n+1.

# (Some) Peano Axioms

Important properties

(1) Different natural numbers have different successors:

$\forall n,m\ [n,m \in \mathbb{N} : s(m) = s(n) \Rightarrow m = n]$

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s is injective!

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stated positively

s is injective!

(2) 0 is not a successor: $\forall n\ [n \in \mathbb{N} : \neg\ (s(n) = 0)\ ]$

(3) All natural numbers except 0 are successors:

$\forall n[n \in \mathbb{N} \wedge \neg(n = 0) : \exists m[m \in \mathbb{N} :\ n = s(m)]$

# There is more to it - induction

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If we know that

1. $D_0$ falls
2. The dominos are close enough together so that if $D_i$ falls, then $D_{i+1}$ falls (for all $i \in \mathbb{N}$)

Then we can conclude that every domino $D_n$ ($n \in \mathbb{N}$) falls!

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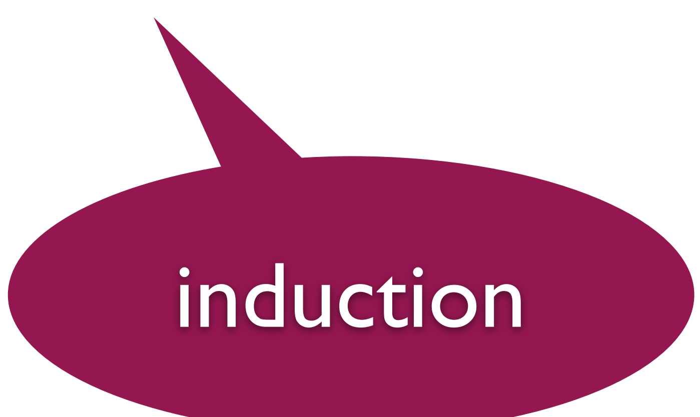

# Induction

$$P(0) \wedge \forall i\ [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n\ [n \in \mathbb{N} : P(n)]$$

# Induction

P - unary predicate over $\mathbb{N}$

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∀ elim with 0

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Variant of the Peano Axiom:

Let $K \subseteq \mathbb{N}$ have the property that

(a) $0 \in K$ and

(b) for all $n \in \mathbb{N}$, $n \in K \Rightarrow (n+1) \in K$.

Then $K = \mathbb{N}$.

# Induction

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(m) $P(0)$

{Assume}

(k) **var** $i$; $i \in \mathbb{N}$

(k+1) $P(i)$

...

(l-1) $P(i+1)$

{$\Rightarrow$-intro on (k+1) and (l-1)}

(l) $P(i) \Rightarrow P(i+1)$

{$\forall$-intro on (k) and (l)}

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Basis

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(k+1) $P(i)$ — induction hypothesis

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Inductive proof: truth is passed on

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**Example**

The sequence of real numbers $(a_i \mid i \in \mathbb{N})$ is defined inductively by

$$a_0 = 2$$
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|---|---|---|---|---|---|
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**Conjecture**

For all $n \in \mathbb{N}$ it holds that

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proof by induction

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∀ elim with k=1

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⋮

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Definition of

$(a_i \mid i \in \mathbb{N})$

with strong induction

$a_n$ is defined via

$a_0, .., a_{n-1}$

# Cardinality

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**Def.** Two sets A and B have the same cardinality (are equinumerous) if there is a bijection $f: A \to B$. Notation $A \sim B$, or $|A| = |B|$.

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**Theorem (Cantor)**

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

# Operations on cardinals

$|A| = [A]_{\sim}$

cardinal numbers are ~ equivalence classes

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Def.

Let A and B be two disjoint sets. Then
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Note: $2 = |\{0,1\}|$

# Finite sets, finite cardinals

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# Finite sets, finite cardinals

We write $\mathbb{N}_k$ for the set $\{0,1,..., k-1\}$. Then $\mathbb{N}_0 = \varnothing$.

We will also write k for $|\mathbb{N}_k|$.

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This justifies the notation.

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E.g. If $|A| = k$ and $|B| = m$ for some $k,m \in \mathbb{N}$ then $|A \times B| = k \cdot m$

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# Infinite, countable and uncountable sets

Time for a video!

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For every set A we have $|A| < |\mathcal{P}(A)|$.

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**Theorem (Cantor)**

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Hence, for every cardinal there is a larger one.