Sets

- A set S is a collection of different objects, the elements of S
- We write $x \in S$ for `x is an element of S'
- A set `can' be specified by
  (1) listing its elements, e.g. $S = \{1,3,7,18\}$
  (2) specifying a property, e.g. $S = \{x \mid P(x)\}$

P is a proposition over x, which is true or false

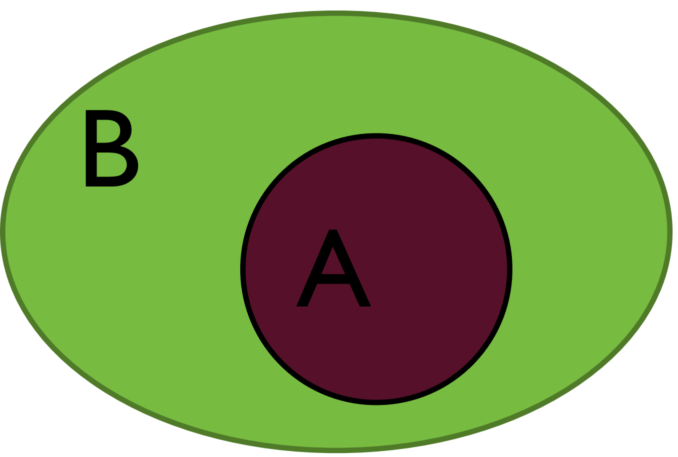
- Sets can be finite e.g. $\{\clubsuit, \heartsuit\}$ or infinite e.g. $\mathbb{N}$
- The set with no elements is the empty set, notation $\varnothing$
- The `number' of elements in a set S is the cardinality of S, notation $|S|$

Sets - properties

- All elements of a set are different
- The elements of a set are not ordered
- The same set can be specified in different ways, e.g. {1,2,3,4}, {2,3,1,4}, $\{i \mid i \in \mathbb{N} \text{ and } 0 < i < 5\}$

Subsets, equality

Def. $A \subseteq B$ iff all elements of A are elements of B



[iff for all a, if $a \in A$, then $a \in B$]

[iff $\forall a \, (a \in A \Rightarrow a \in B)$]

logical formula

quantifier

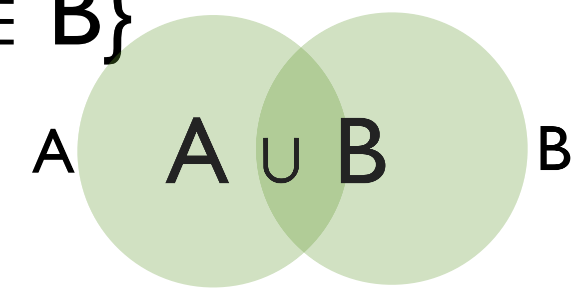
logical connective

Def. $A = B$ iff $A \subseteq B$ and $B \subseteq A$

Def. $A \subset B$ iff $A \subseteq B$ and $A \neq B$

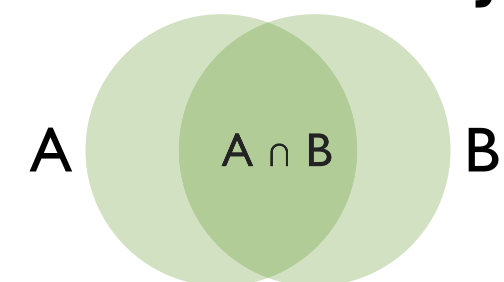
Operations on sets

Def. Union (Vereinigung) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

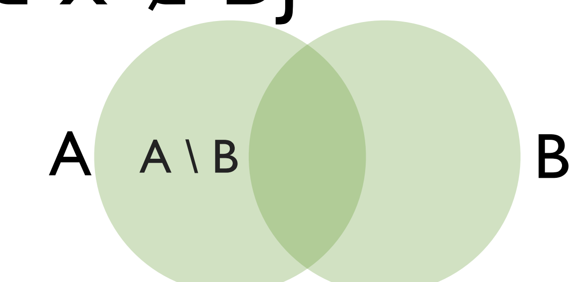


Def. Intersection (Durchschnitt) $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

A and B are disjoint if $A \cap B = \varnothing$



Def. Difference (Differenz) $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



Def. Direct product (Kartesisches Produkt)

$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$

ordered pairs

$(A \times B) \times C \neq A \times (B \times C)$

Def. Powerset (Potenzmenge) $\mathcal{P}(A) = \{X \mid X \subseteq A\}$

Russell's paradox

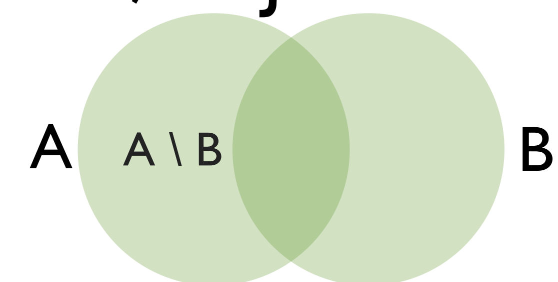
- Let P be the set of all sets that are not an element of itself
- Hence, $P = \{ x \mid x \notin x\}$
- Is $P \in P$?
- Contradiction!

The need for a universal set U

$$S = \{x \mid x \in U \text{ and } P(x)\}$$

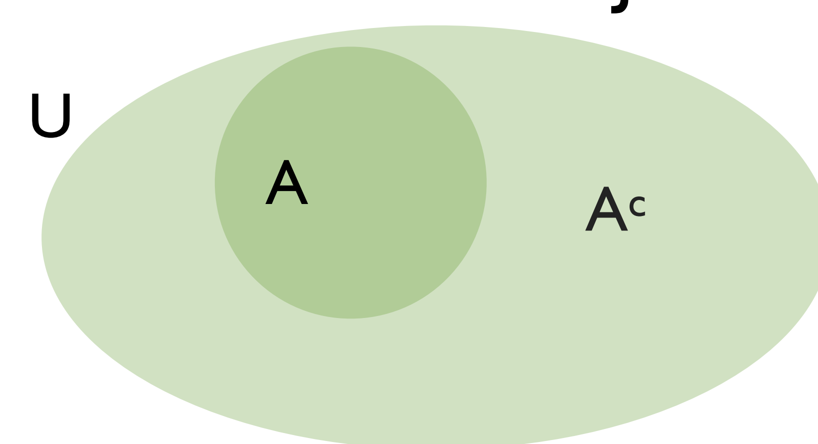
Operations on sets

Def. Difference (Differenz) $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



Given a universal set U

Def. Complement (Komplement) $A^c = \{x \mid x \in U \text{ and } x \notin A\}$



Hence $A^c = U \setminus A$

Properties of sets

1.	$\varnothing \subseteq X$
2.	If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
3.	$X \cap Y \subseteq X$, $X \cap Y \subseteq Y$
4.	$X \subseteq X \cup Y$, $Y \subseteq X \cup Y$
5.	If $X' \subseteq Y'$ and $X'' \subseteq Y''$, then $X' \cap X'' \subseteq Y' \cap Y''$
6.	If $X' \subseteq Y'$ and $X'' \subseteq Y''$, then $X' \cup X'' \subseteq Y' \cup Y''$
7.	$X \cap Y = X$ iff $X \subseteq Y$
8.	$X \cap X = X$ (idempotence)
9.	$X \cup X = X$ (idempotence)
10.	$X \cap \varnothing = \varnothing$

Properties of sets

11.	$X \cup \emptyset = X$
12.	$X \cap Y = Y \cap X$ (commutativity)
13.	$X \cup Y = Y \cup X$ (commutativity)
14.	$X \cap (Y \cap Z) = (X \cap Y) \cap Z$ (associativity)
15.	$X \cup (Y \cup Z) = (X \cup Y) \cup Z$ (associativity)
16.	$X \cap (X \cup Y) = X$ (absorption)
17.	$X \cup (X \cap Y) = X$ (absorption)
18.	$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ (distributivity)
19.	$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ (distributivity)
20.	$X \setminus Y \subseteq X$

Properties of sets

21.	$(X \setminus Y) \cap Y = \emptyset$
22.	$X \cup Y = X \cup (Y \setminus X)$
23.	$X \setminus X = \emptyset$
24.	$X \setminus \emptyset = X$
25.	$\emptyset \setminus X = \emptyset$
26.	If $X \subseteq Y$, then $X \setminus Y = \emptyset$
27.	$(X^c)^c = X$
28.	$(X \cap Y)^c = X^c \cup Y^c$ (De Morgan)
29.	$(X \cup Y)^c = X^c \cap Y^c$ (De Morgan)
30.	$X \times \emptyset = \emptyset$ $\emptyset \times X = \emptyset$
31.	$\emptyset \times X = \emptyset$
32.	If $X \subseteq Y$, then $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$