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Naive Set Theory  
Relations

# Product of multiple sets

Direct product (Kartesisches Produkt)

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

ordered pairs

$$(A \times B) \times C \neq A \times (B \times C)$$

Therefore, we define

$$A \times B \times C = \{(x, y, z) \mid x \in A \text{ and } y \in B \text{ and } z \in C\}$$

if  $A_i = A$  for all  $i$ ,  
then the product is  
denoted  $A^n$

sequence of  
length  $n$

In general, for sets  $A_1, A_2, \dots, A_n$  with  $n \geq 1$ ,

$$A_1 \times A_2 \times \dots \times A_n = \prod_{1 \leq i \leq n} A_i = \{(x_1, x_2, \dots, x_n) \mid x_i \in A_i \text{ for } 1 \leq i \leq n\}$$

# Relations

**Def.** If  $A$  and  $B$  are sets, then any subset  $R \subseteq A \times B$  is a (binary) relation between  $A$  and  $B$

similarly, unary relation (subset), n-ary relation...

**Def.**  $R$  is a relation on  $A$  if  $R \subseteq A \times A$

some relations are special

# Special relations

A relation  $R \subseteq A \times A$  is:

reflexive	iff	for all $a \in A$ , $(a,a) \in R$
symmetric	iff	for all $a,b \in A$ , if $(a,b) \in R$ , then $(b,a) \in R$
transitive	iff	for all $a,b,c \in A$ , if $(a,b) \in R$ and $(b,c) \in R$ , then $(a,c) \in R$
irreflexive	iff	for all $a \in A$ , $(a,a) \notin R$
antisymmetric	iff	for all $a,b \in A$ , if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$
asymmetric	iff	for all $a,b \in A$ , if $(a,b) \in R$ , then $(b,a) \notin R$
total	iff	for all $a,b \in A$ , $(a,b) \in R$ or $(b,a) \in R$

(infix) notation  $aRb$  for  $(a,b) \in R$

# Special relations

A relation  $R$  on  $A$ , i.e.,  $R \subseteq A \times A$  is:

- equivalence      iff       $R$  is reflexive, symmetric, and transitive
- partial order    iff       $R$  is reflexive, antisymmetric, and transitive
- strict order     iff       $R$  is irreflexive and transitive
- preorder        iff       $R$  is reflexive and transitive
- total (linear)  
order            iff       $R$  is a total partial order

# Obvious properties

1. Every partial order is a preorder.
2. Every total order is a partial order.
3. Every total order is a preorder.
4. If  $R \subseteq A \times A$  is a relation such that there are  $a, b \in A$  with  
 $a \neq b, (a,b) \in R$  and  $(b,a) \in R$ ,  
then  $R$  is not a partial order, nor a total order, nor a strict order.

# Operations on relations

Let  $R \subseteq A \times \underline{B}$  and  $S \subseteq \underline{B} \times C$  be two relations. Their composition is the relation

$$R \circ S = \{(a,c) \in A \times C \mid \text{there is } b \in B \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\}$$

relational composition is associative  $(R \circ S) \circ T = R \circ (S \circ T)$

so again we write  
 $R^n = \underbrace{R \circ R \circ \dots \circ R}_{n \text{ times}}$

Let  $R \subseteq A \times B$  be a relation. The inverse relation of  $R$  is the relation

$$R^{-1} = \{(b,a) \in B \times A \mid (a,b) \in R\}$$

# Characterizations

**Lemma:** Let  $R$  be a relation over the set  $A$ . Then

1.  $R$  is reflexive      iff  $\Delta_A \subseteq R$
2.  $R$  is symmetric    iff  $R \subseteq R^{-1}$
3.  $R$  is transitive     iff  $R^2 \subseteq R$



# Important equivalence on $\mathbb{Z}$

**Def.** For a natural number  $n$ , the relation  $\equiv_n$  is defined as

$$i \equiv_n j \quad \text{iff } n \mid i - j$$

[iff  $i-j$  is a multiple of  $n$  ]

[iff there exists  $k \in \mathbb{Z}$  s.t.  $i-j = k \cdot n$  ]

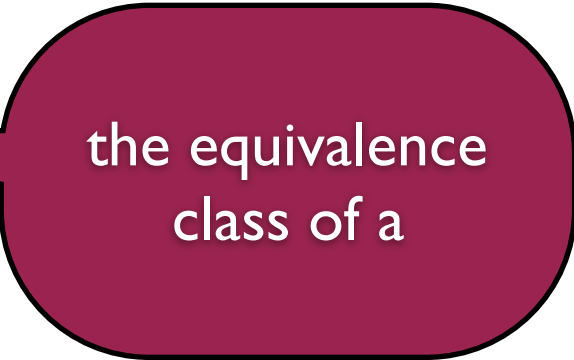
[iff  $\exists k (k \in \mathbb{Z} \wedge i-j = k \cdot n)$  ]

**Lemma:** The relation  $\equiv_n$  is an equivalence for every  $n$ .

# Equivalences classes

**Def.** Let  $R$  be an equivalence over  $A$  and  $a \in A$ . Then

$$[a]_R = \{ b \in A \mid (a, b) \in R \}$$



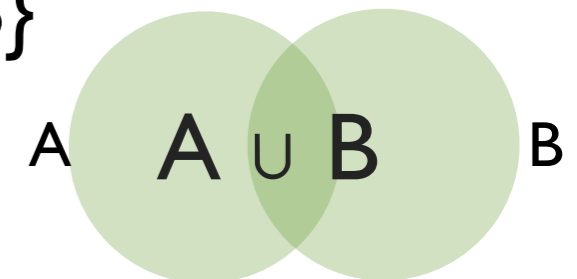
the equivalence  
class of  $a$

**Lemma E1:** Let  $R$  be an equivalence over the set  $A$ . Then  
for all  $a, b \in A$ ,  $[a]_R = [b]_R$  or  $[a]_R \cap [b]_R = \emptyset$

**Task:** Describe the equivalence classes of  $\equiv_n$   
How many classes are there?

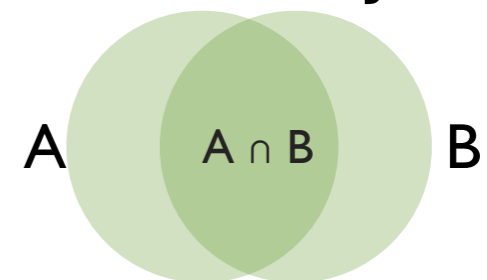
# Unions and intersections of multiple sets

Union (**Vereinigung**)  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



Intersection (**Durchschnitt**)  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

A and B are **disjoint** if  $A \cap B = \emptyset$



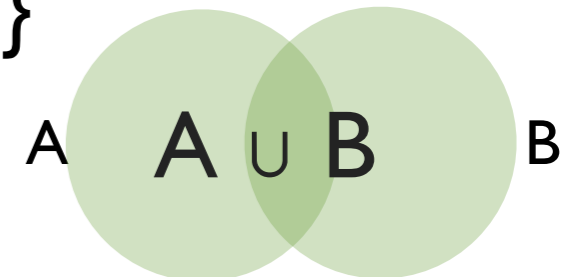
In general, for sets  $A_1, A_2, \dots, A_n$  with  $n \geq 1$ ,

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{1 \leq i \leq n} A_i = \{x \mid x \in A_i \text{ for some } i \in \{1, \dots, n\}\}$$

$$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{1 \leq i \leq n} A_i = \{x \mid x \in A_i \text{ for all } i \in \{1, \dots, n\}\}$$

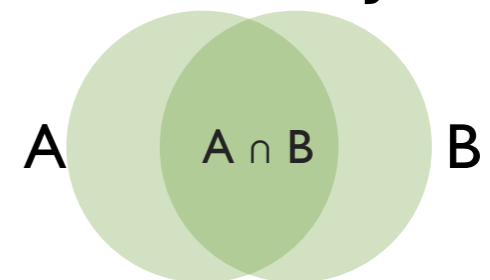
# Unions and intersections of multiple sets

Union (**Vereinigung**)  $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



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In general, for a **family of sets**  $(A_i \mid i \in I)$

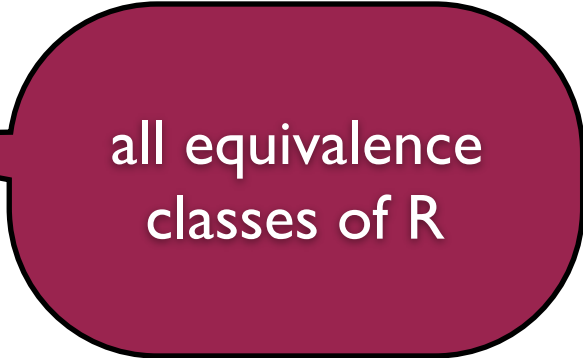
$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

# Back to equivalence classes

**Example:** Let  $R$  be an equivalence over  $A$  and  $a \in A$ . Then

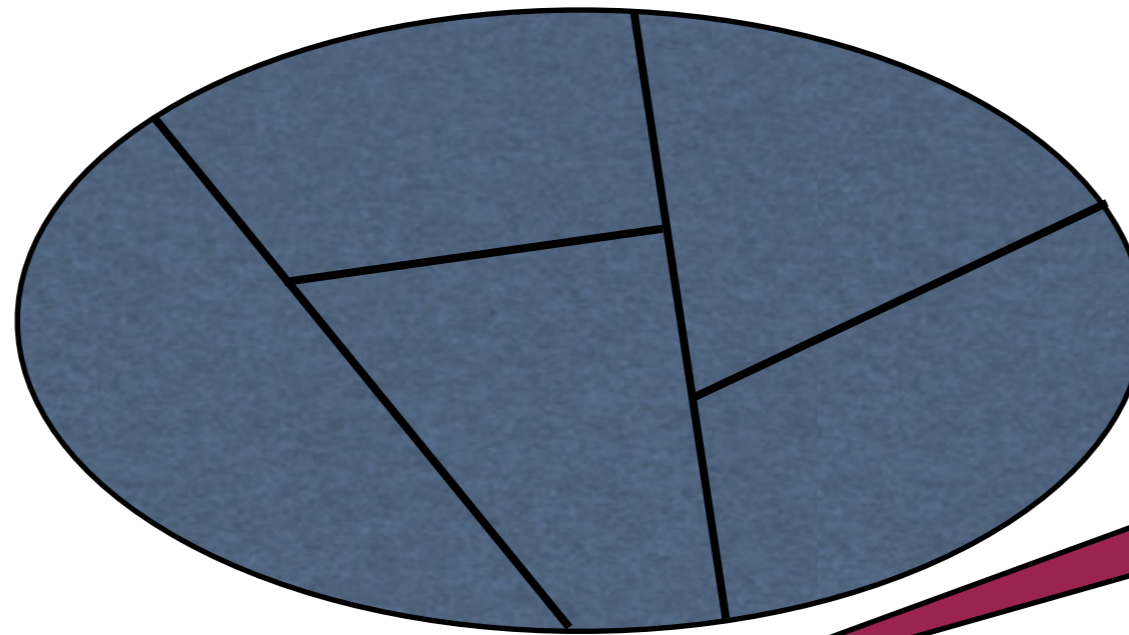
$( [a]_R, a \in A )$  is a family of sets.



all equivalence classes of  $R$

**Lemma E2:**  $A = \bigcup_{a \in A} [a]_R$ . The union is disjoint.

# Partitions



hence, a collection  
of  
subsets of  $X$

**Def.** Let  $X$  be a set. A subset  $P$  of the powerset  $\mathcal{P}(X)$  is a partition (**Klasseneinteilung**) of  $X$  if it satisfies:

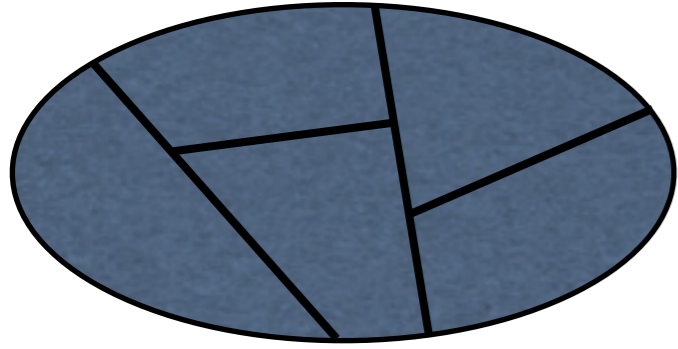
(1) For all  $A \in P$ ,  $A \neq \emptyset$

(2) For all  $A, B \in P$ , if  $A \neq B$

then  $A \cap B = \emptyset$

(3)  $\bigcup_{A \in P} A = X$

that are non-empty,  
pairwise disjoint,  
and their union equals  $X$



# Partitions = Equivalences

**Theorem PE:** Let  $X$  be a set.

(1) If  $R$  is an equivalence on  $X$ , then the set

$$P(R) = \{ [x]_R \mid x \in X \}$$

is a partition of  $X$ .

(2) If  $P$  is a partition of  $X$ , then the relation

$$R(P) = \{ (x, y) \in X \times X \mid \text{there is } A \in P \text{ such that } x, y \in A \}$$

is an equivalence relation.

Moreover, the assignments  $R \mapsto P(R)$  and  $P \mapsto R(P)$  are inverse to each other, i.e.,  $R(P(R)) = R$  and  $P(R(P)) = P$ .

# Transitive closure

Let  $R$  be a relation on a set  $X$ . The transitive closure (**transitive Hülle**) of  $R$ , notation  $R^+$ , is the relation

$$R^+ = \bigcup_{n \in \mathbb{N}, n \neq 0} R^n$$


$$R^{n+1} = R^n \circ R$$

The reflexive and transitive closure (**reflexive und transitive Hülle**) of  $R$ , notation  $R^*$ , is the relation

$$R^* = \bigcup_{n \in \mathbb{N}} R^n$$


$$R^0 = \Delta_R$$

**Proposition TC:** Let  $R$  be a relation on  $X$ . The transitive closure of  $R$  is the smallest transitive relation that contains  $R$ . The reflexive and transitive closure of  $R$  is the smallest reflexive and transitive relation that contains  $R$ .