The structure of natural numbers

is helpful for proving properties

$\forall n[n \in \mathbb{N}: P(n)]$

The structure of natural numbers

On natural numbers we can define a notion of a successor, a mapping

s: $\mathbb{N} \to \mathbb{N}$

by s(n) = n+1

The successor mapping imposes a structure on the set that enables us to count:

1) there is a starting natural number 0
2) for every natural number n, there is a next natural number s(n) = n+1.

(Some) Peano Axioms

Important properties

(1) Different natural numbers have different successors:

$\forall n,m\ [n,m \in \mathbb{N} : s(m) = s(n) \Rightarrow m = n]$

stated positively

s is injective!

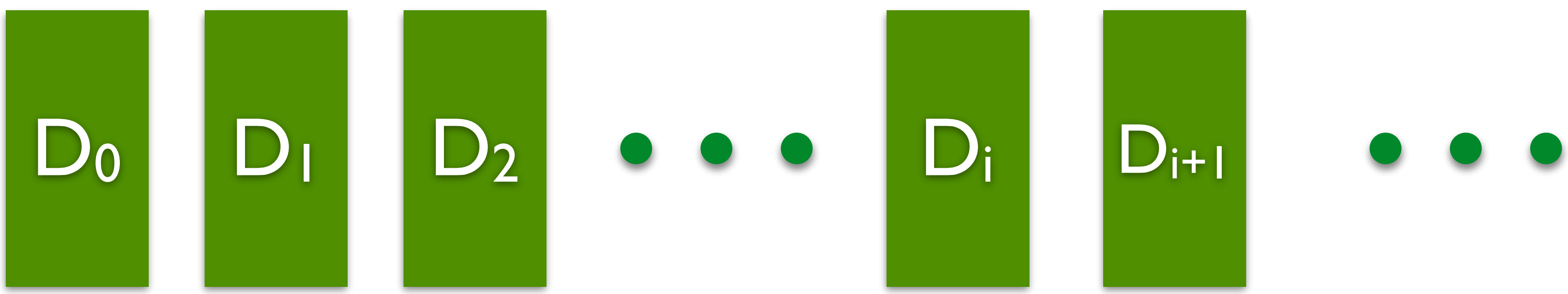
(2) 0 is not a successor: $\forall n\ [n \in \mathbb{N} : \neg\ (s(n) = 0)\ ]$

(3) All natural numbers except 0 are successors:

$\forall n[n \in \mathbb{N} \wedge \neg(n = 0) : \exists m[m \in \mathbb{N} :\ n = s(m)]$

There is more to it - induction

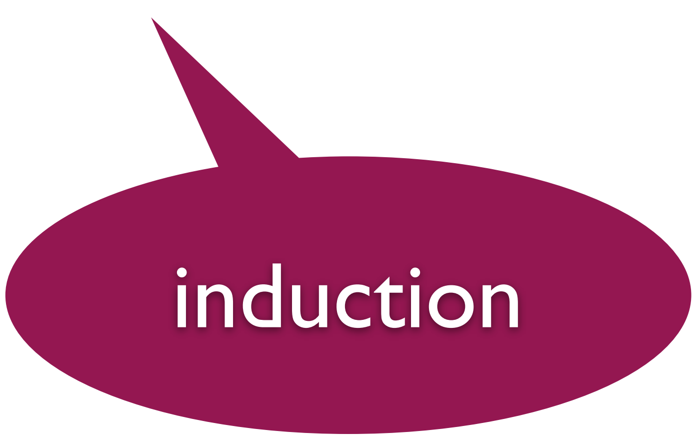
Imagine an infinite sequence of dominos



If we know that

1. D_0 falls
2. The dominos are close enough together so that if D_i falls, then D_{i+1} falls (for all $i \in \mathbb{N}$)

Then we can conclude that every domino D_n ($n \in \mathbb{N}$) falls!



Induction

P - unary predicate over $\mathbb{N}$

$$P(0) \wedge \forall i\, [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n\, [n \in \mathbb{N} : P(n)]$$

∀ elim with 0

⇒ elim

$P(0)$

$P(0) \Rightarrow P(1)$

$P(1)$

$P(1) \Rightarrow P(2)$

$P(2)$

$P(2) \Rightarrow P(3)$

...

Variant of the Peano Axiom:

Let $K \subseteq \mathbb{N}$ have the property that

(a) $0 \in K$ and

(b) for all $n \in \mathbb{N}$, $n \in K \Rightarrow (n+1) \in K$.

Then $K = \mathbb{N}$.

Induction

$$P(0) \wedge \forall i\,[i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \;\Rightarrow\; \forall n\,[n \in \mathbb{N} : P(n)]$$

P - unary predicate over $\mathbb{N}$

$\ldots$

(m) $P(0)$

{Assume}

(k) **var** $i;\ i \in \mathbb{N}$

(k+1) $P(i)$

$\ldots$

(l-1) $P(i+1)$

{$\Rightarrow$-intro on (k+1) and (l-1)}

(l) $P(i) \Rightarrow P(i+1)$

{$\forall$-intro on (k) and (l)}

(l+1) $\forall i[i \in \mathbb{N} : P(i) \Rightarrow P(i+1)]$

{induction on (m) and (l+1)}

(l+2) $\forall n[n \in \mathbb{N} : P(n)]$

Basis

induction hypothesis

Induction step

Inductive definitions

Inductive proof: truth is passed on

Inductive definition: construction is passed on

well defined by induction

Example

The sequence of real numbers $(a_i \mid i \in \mathbb{N})$ is defined inductively by

$$a_0 = 2$$
$$a_{i+1} = 2a_i - 1$$

a_0	a_1	a_2	a_3	a_4	...
2	3	5	9	17	...

proof by induction

Conjecture

For all $n \in \mathbb{N}$ it holds that

$$a_n = 2^n + 1$$

Strong induction

P - unary predicate over $\mathbb{N}$

$$\forall k\, [k \in \mathbb{N} : \forall j [j \in \mathbb{N} \wedge j < k : P(j)] \Rightarrow P(k)] \Rightarrow \forall n\, [n \in \mathbb{N} : P(n)]$$

$\forall$ elim with k=1

$\Rightarrow$ elim,

$\wedge$ intro

⋮

$P(0)$

$P(0) \Rightarrow P(1)$

$P(0) \wedge P(1)$

$P(0) \wedge P(1) \Rightarrow P(2)$

$P(0) \wedge P(1) \wedge P(2)$

$P(0) \wedge P(1) \wedge P(2) \Rightarrow P(3)$

...

Definition of $(a_i \mid i \in \mathbb{N})$ with strong induction

a_n is defined via $a_0, .., a_{n-1}$