Functions, mappings

Def. If A and B are sets, a function (mapping, Abbildung) f from A to B, notation f: $A \longrightarrow B$ is an assignment (of elements of B to elements of A, we write f(a) for the element assigned to a) s. t.

for every $a \in A$, there exists a unique $b \in B$ such that $b = f(a)$.

Functions, mappings

Def. If A and B are sets, a function (mapping, Abbildung) f from A to B, notation $f\colon A \longrightarrow B$ is an assignment (of elements of B to elements of A, we write f(a) for the element assigned to a) s. t.

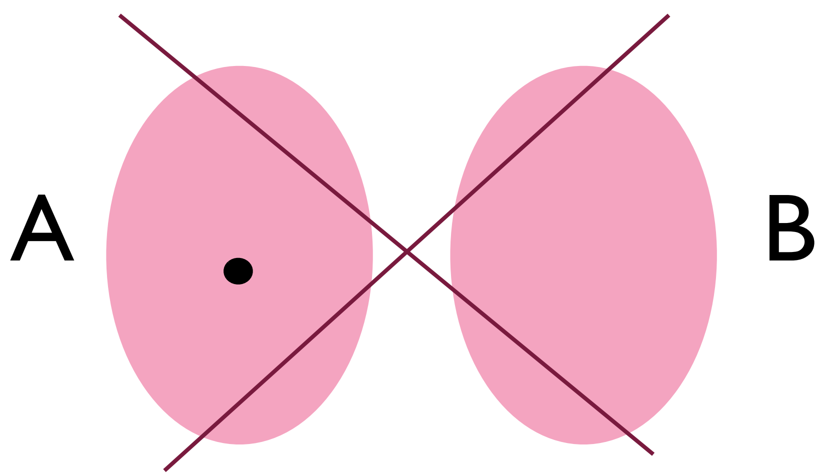
for every $a \in A$, there exists a unique $b \in B$ such that $b = f(a)$.

the image of a,
$b = f(a)$
$a \mapsto b$

Functions, mappings

Def. If A and B are sets, a function (mapping, Abbildung) f from A to B, notation f: $A \longrightarrow B$ is an assignment (of elements of B to elements of A, we write f(a) for the element assigned to a) s. t.

for every $a \in A$, there exists a unique $b \in B$ such that $b = f(a)$.

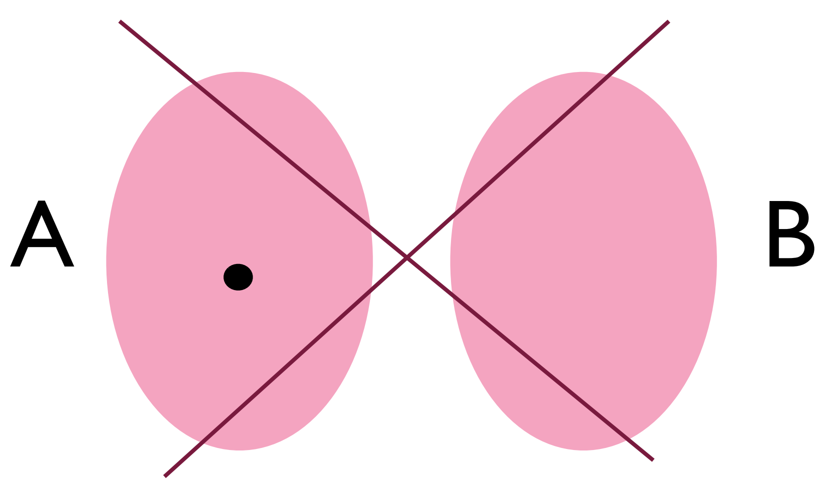


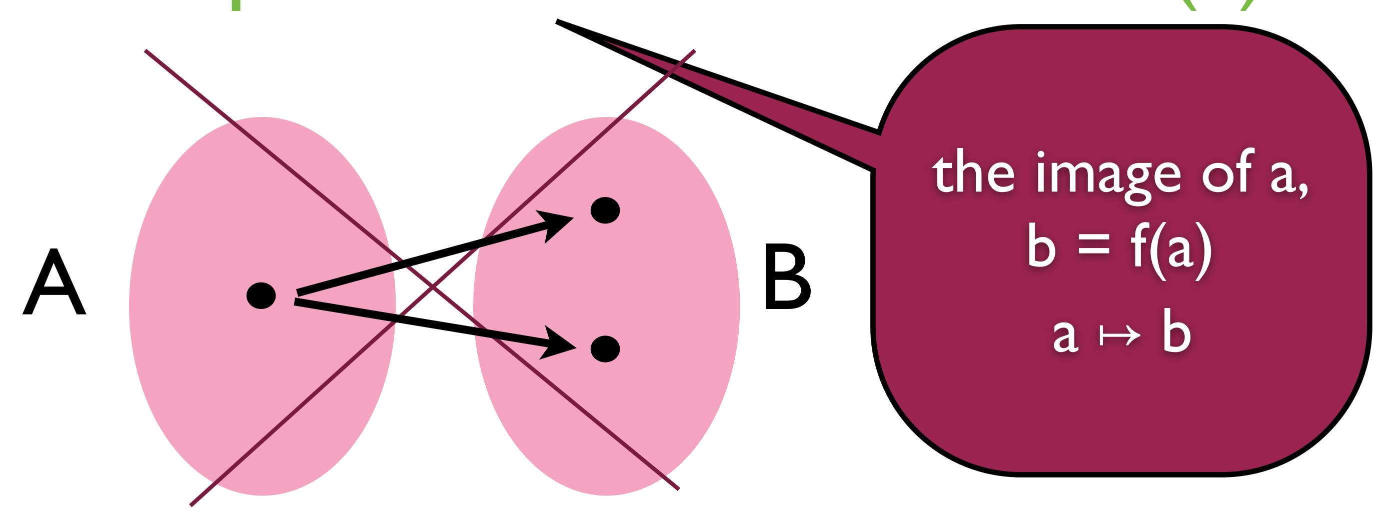
the image of a,
$b = f(a)$
$a \mapsto b$

Functions, mappings

Def. If A and B are sets, a function (mapping, Abbildung) f from A to B, notation $f\colon A \longrightarrow B$ is an assignment (of elements of B to elements of A, we write f(a) for the element assigned to a) s. t.

for every $a \in A$, there exists a unique $b \in B$ such that $b = f(a)$.

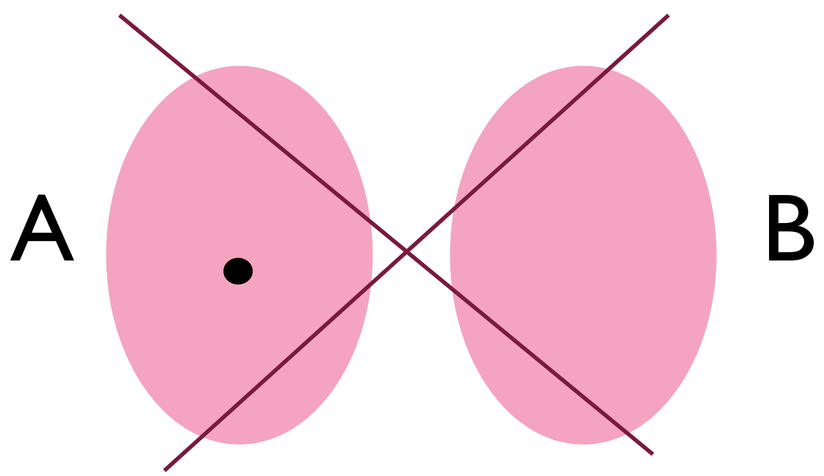


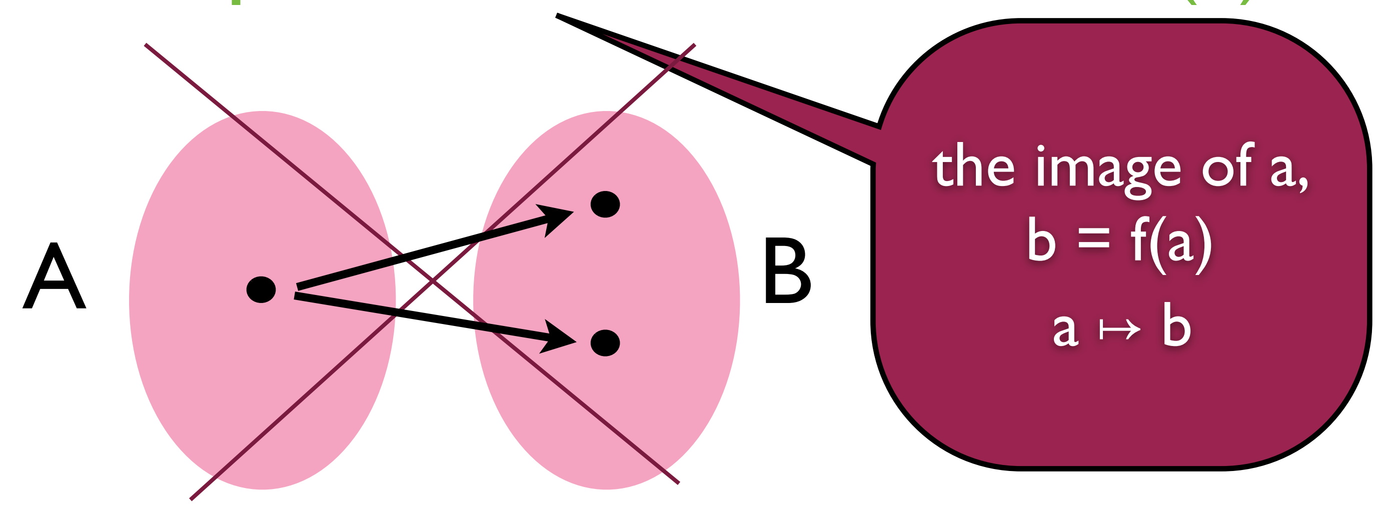


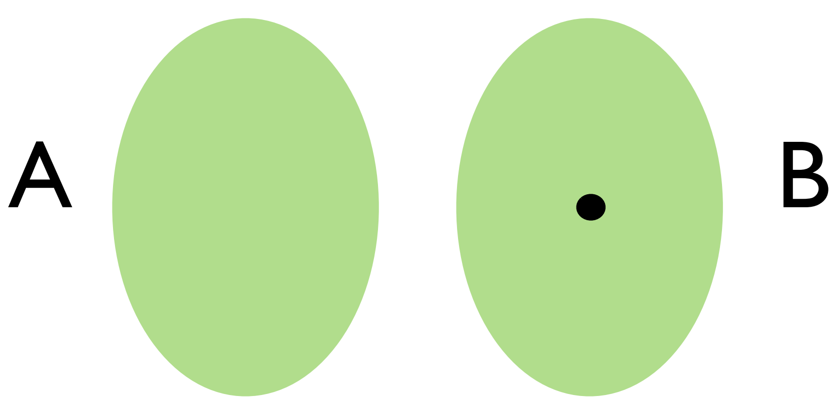
Functions, mappings

Def. If A and B are sets, a function (mapping, Abbildung) f from A to B, notation $f: A \longrightarrow B$ is an assignment (of elements of B to elements of A, we write f(a) for the element assigned to a) s. t.

for every $a \in A$, there exists a unique $b \in B$ such that $b = f(a)$.



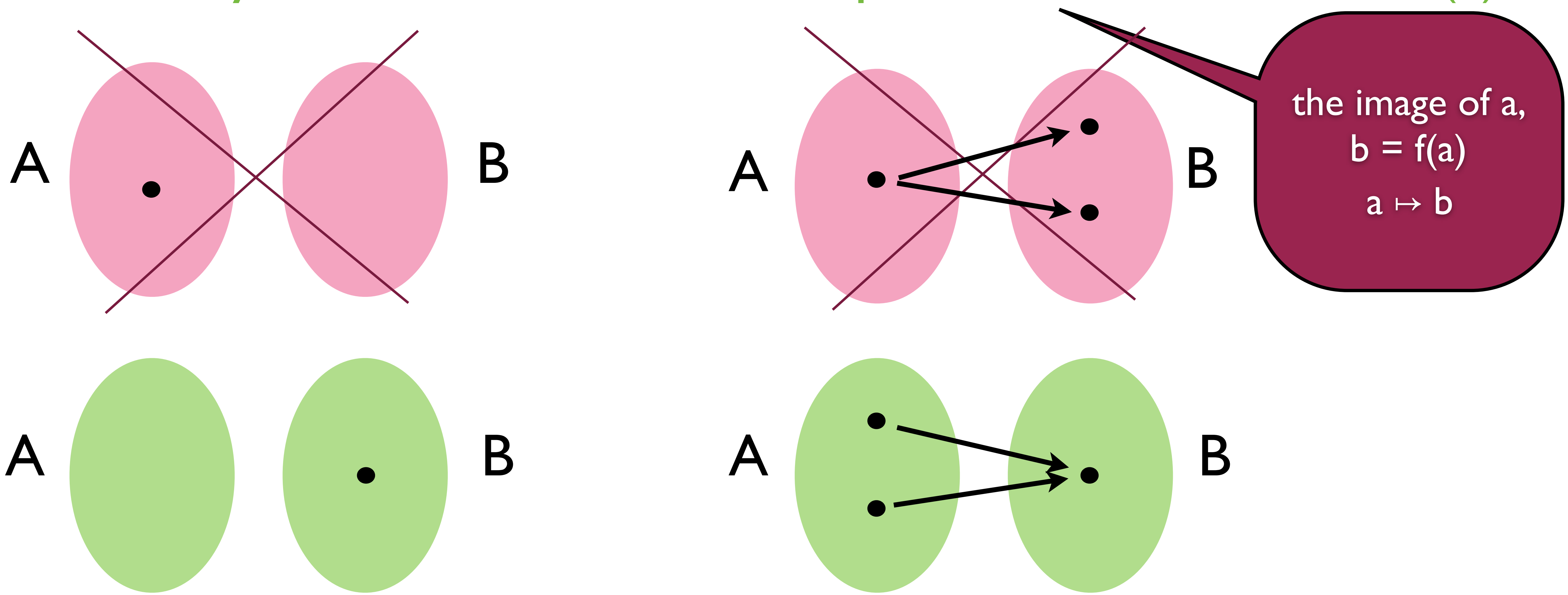




Functions, mappings

Def. If A and B are sets, a function (mapping, Abbildung) f from A to B, notation $f\colon A \longrightarrow B$ is an assignment (of elements of B to elements of A, we write f(a) for the element assigned to a) s. t.

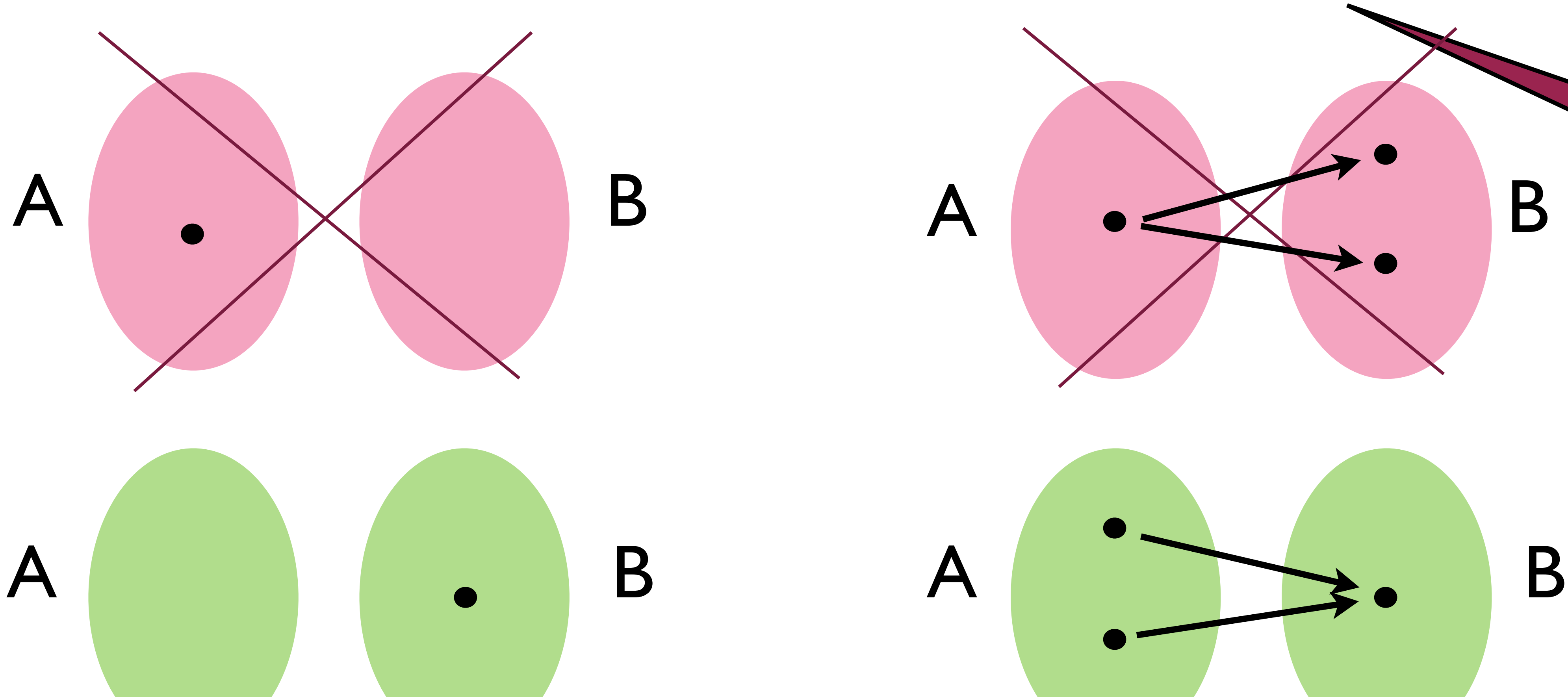
for every $a \in A$, there exists a unique $b \in B$ such that $b = f(a)$.



Functions, mappings

Def. If A and B are sets, a function (mapping, Abbildung) f from A to B, notation $f: A \longrightarrow B$ is an assignment (of elements of B to elements of A, we write f(a) for the element assigned to a) s. t.

for every $a \in A$, there exists a unique $b \in B$ such that $b = f(a)$.



the image of a,
$b = f(a)$
$a \mapsto b$

$\{(a, f(a)) \mid a \in A\}$ is the graph of the function f

Functions, mappings

When f: A $\longrightarrow$ B then dom f = A and cod f = B

Functions, mappings

When f: A $\longrightarrow$ B then dom f = A and cod f = B

domain of F
(Definitionsbereich)

Functions, mappings

When $f: A \longrightarrow B$ then $\mathrm{dom}\ f = A$ and $\mathrm{cod}\ f = B$

domain of F
(Definitionsbereich)

codomain of F
(Wertebereich)

Functions, mappings

When f: $A \longrightarrow B$ then dom f = A and cod f = B

domain of F
(Definitionsbereich)

codomain of F
(Wertebereich)

Let f: $A \longrightarrow B$ and $A' \subseteq A$.

The image (Bild) of A' is the set $f(A') = \{f(a) \mid a \in A'\} \subseteq B$.

Functions, mappings

When f: $A \longrightarrow B$ then dom f = A and cod f = B

domain of F
(Definitionsbereich)

codomain of F
(Wertebereich)

Let f: $A \longrightarrow B$ and $A' \subseteq A$.

The image (Bild) of A' is the set $f(A') = \{f(a) \mid a \in A'\} \subseteq B$.

if $a \in A'$, then $f(a) \in f(A')$

Functions, mappings

When f: A $\longrightarrow$ B then dom f = A and cod f = B

domain of F (Definitionsbereich)

codomain of F (Wertebereich)

Let f: A $\longrightarrow$ B and A' $\subseteq$ A.

The image (Bild) of A' is the set f(A') = {f(a) | a $\in$ A'} $\subseteq$ B.

f(A') = {b $\in$ B | there is an a $\in$ A' with b = f(a)}

if a $\in$ A', then f(a) $\in$ f(A')

Functions, mappings

When f: $A \longrightarrow B$ then dom f = A and cod f = B

domain of F
(Definitionsbereich)

codomain of F
(Wertebereich)

Let f: $A \longrightarrow B$ and $A' \subseteq A$.

The image (Bild) of A' is the set $f(A') = \{f(a) \mid a \in A'\} \subseteq B$.

$f(A') = \{b \in B \mid \text{there is an } a \in A' \text{ with } b = f(a)\}$

if $a \in A'$, then $f(a) \in f(A')$

So f extends to a function f: $\mathcal{P}(A) \longrightarrow \mathcal{P}(B)$, the image-function.

Functions, mappings

Let $f\colon A \longrightarrow B$ and $B' \subseteq B$.

The inverse image (Urbild) of B' is the set

$$f^{-1}(B') = \{a \mid f(a) \in B'\} \subseteq A.$$

Functions, mappings

Let f: $A \longrightarrow B$ and $B' \subseteq B$.

The inverse image (Urbild) of B' is the set

$$f^{-1}(B') = \{a \mid f(a) \in B'\} \subseteq A.$$

$a \in f^{-1}(B')$ iff $f(a) \in B'$

Functions, mappings

Let f: $A \longrightarrow B$ and $B' \subseteq B$.

The inverse image (Urbild) of B' is the set

$$f^{-1}(B') = \{a \mid f(a) \in B'\} \subseteq A.$$

$a \in f^{-1}(B')$ iff $f(a) \in B'$

Again the inverse image induces a function $f^{-1}: \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$, the inverse-image-function.

Functions, mappings

Let f: $A \longrightarrow B$ and $B' \subseteq B$.

The inverse image (Urbild) of B' is the set

$$f^{-1}(B') = \{a \mid f(a) \in B'\} \subseteq A.$$

$a \in f^{-1}(B')$ iff $f(a) \in B'$

Again the inverse image induces a function $f^{-1}: \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$, the inverse-image-function.

Lemma F1: Let f: $A \longrightarrow B$, $A' \subseteq A$, and $B' \subseteq B$. Then

$$A' \subseteq f^{-1}(f(A')) \quad \text{and} \quad f(f^{-1}(B')) \subseteq B'$$

(in general no more than this holds)

Equality of functions

Let $f: A \longrightarrow B$ and $g: C \longrightarrow D$

Equality of functions

Let $f: A \longrightarrow B$ and $g: C \longrightarrow D$

Def. The functions $f: A \longrightarrow B$ and $g: C \longrightarrow D$ are equal iff

(1) $A = C$
(2) $B = D$
(3) for all $a \in A$, $f(a) = g(a)$.

Equality of functions

Let $f: A \longrightarrow B$ and $g: C \longrightarrow D$

Def. The functions $f: A \longrightarrow B$ and $g: C \longrightarrow D$ are equal iff

(1) $A = C$

(2) $B = D$

(3) for all $a \in A$, $f(a) = g(a)$.

dom f = dom g

Equality of functions

Let $f: A \longrightarrow B$ and $g: C \longrightarrow D$

Def. The functions $f: A \longrightarrow B$ and $g: C \longrightarrow D$ are equal iff

(1) $A = C$

(2) $B = D$

(3) for all $a \in A$, $f(a) = g(a)$.

dom f = dom g

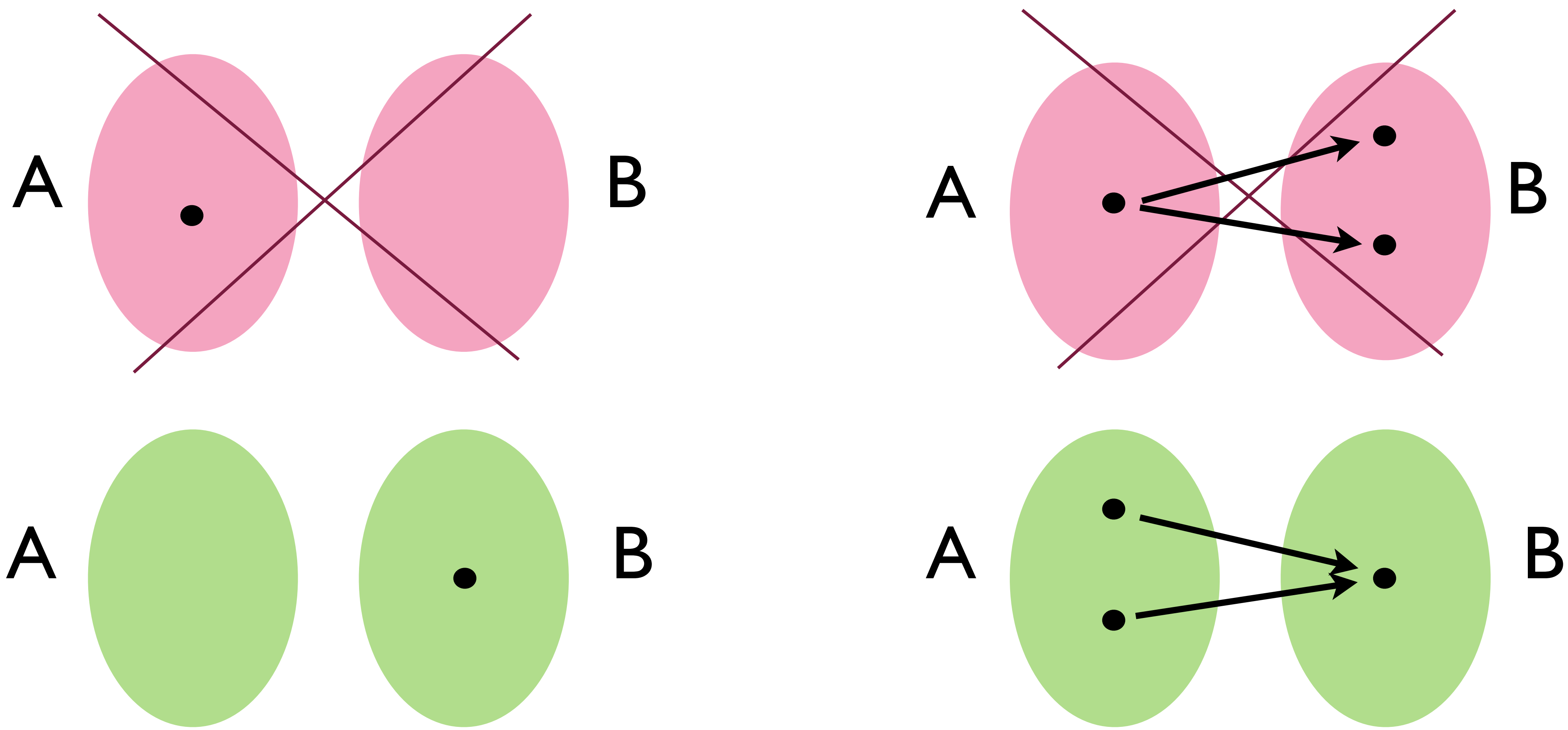
cod f = cod g

Recall...

Def. If A and B are sets, a function f from A to B, notation

$f\colon A \longrightarrow B$ is an assignment s. t.

for every $a \in A$, there exists a unique $b \in B$ such that $b = f(a)$.

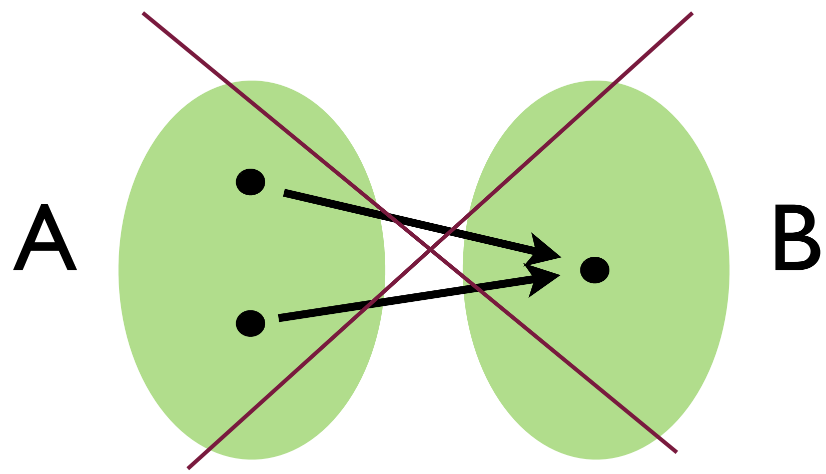


Special functions

The number of ingoing arrows for a function can be 0, 1, or more.
Based on this, we distinguish some special functions.

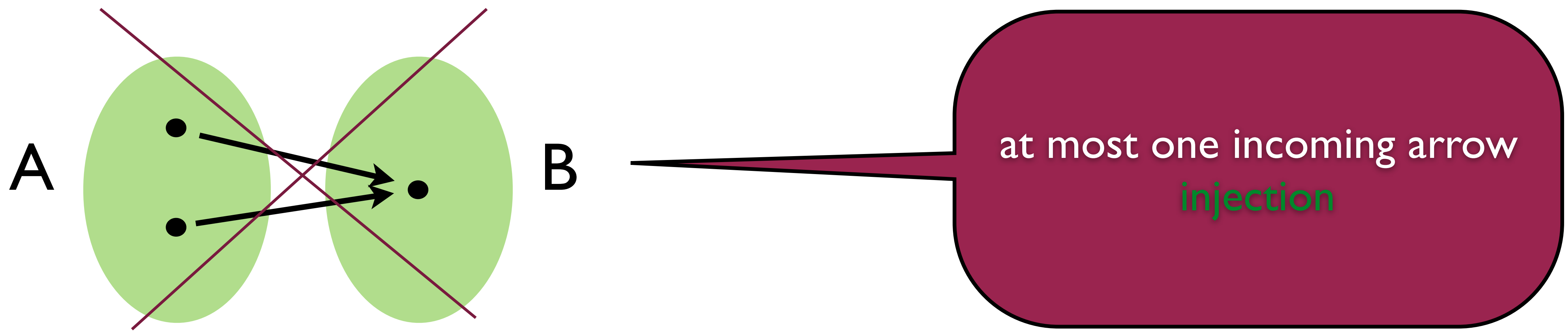
Special functions

The number of ingoing arrows for a function can be 0,1, or more.
Based on this, we distinguish some special functions.



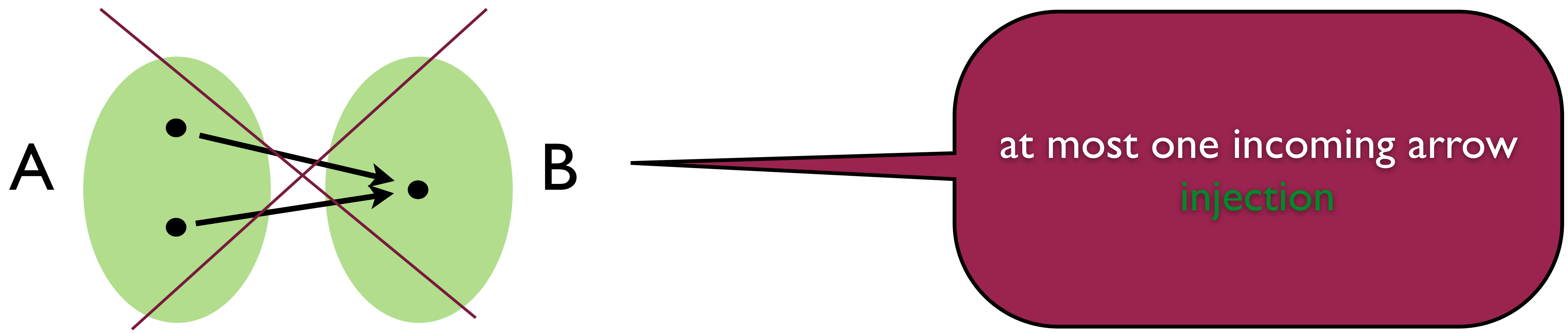
Special functions

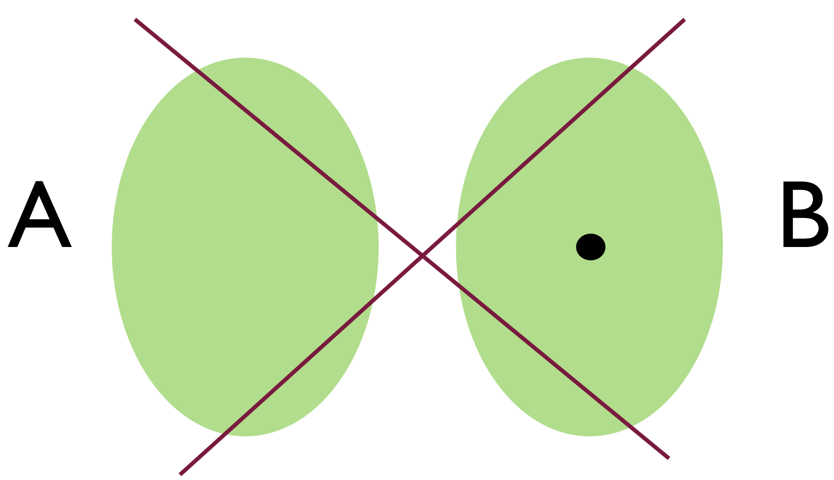
The number of ingoing arrows for a function can be 0,1, or more.
Based on this, we distinguish some special functions.



Special functions

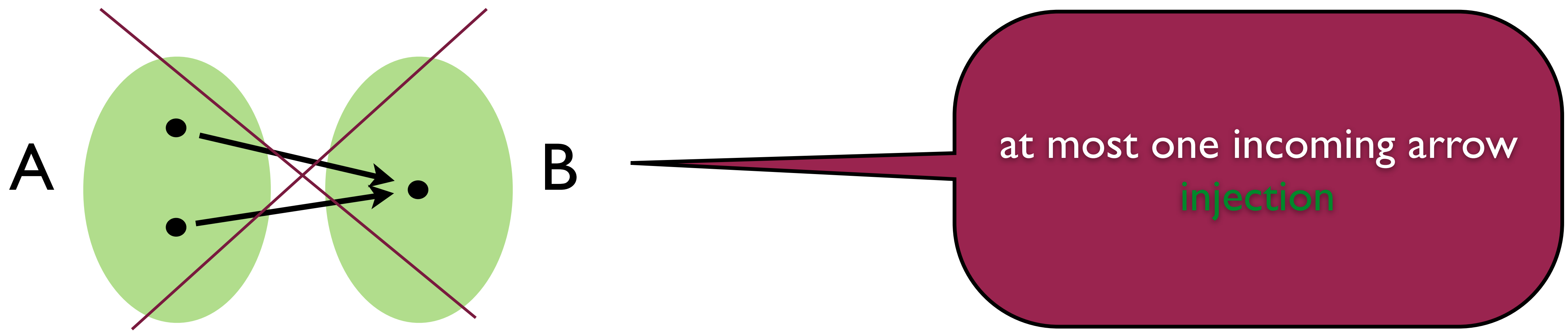
The number of ingoing arrows for a function can be 0,1, or more.
Based on this, we distinguish some special functions.

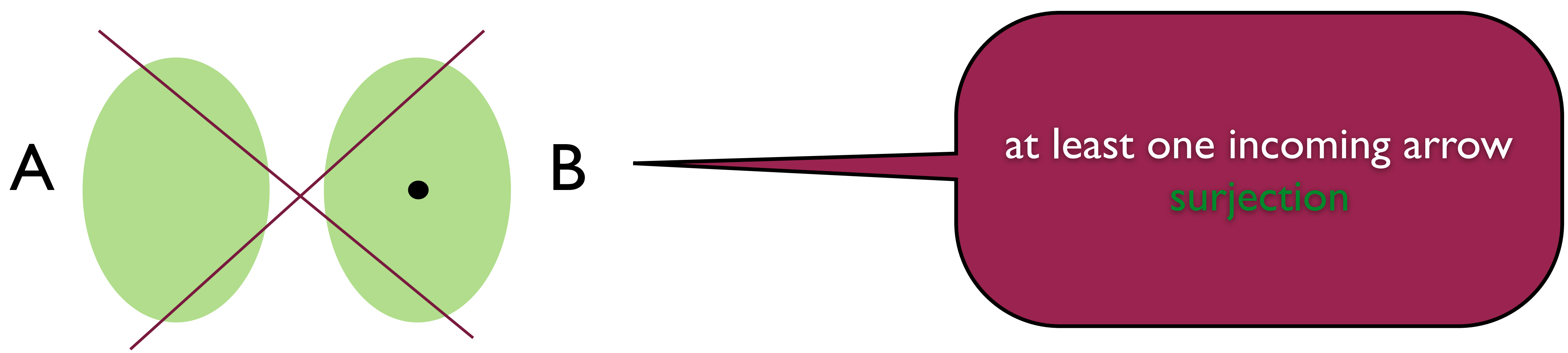




Special functions

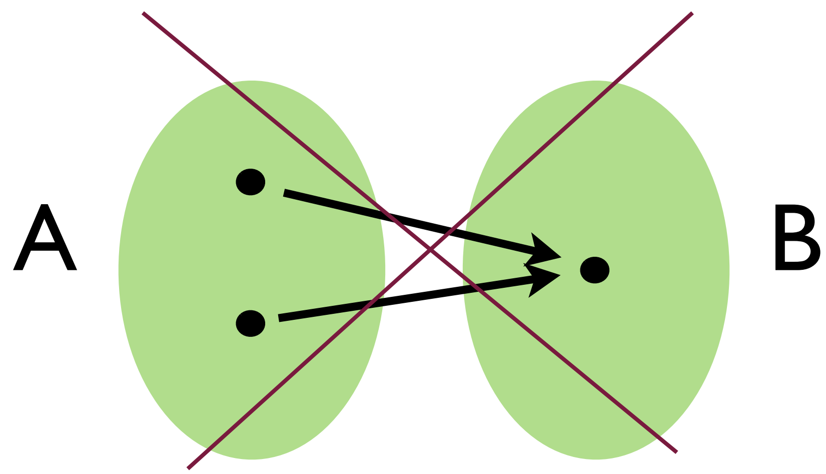
The number of ingoing arrows for a function can be 0,1, or more.
Based on this, we distinguish some special functions.



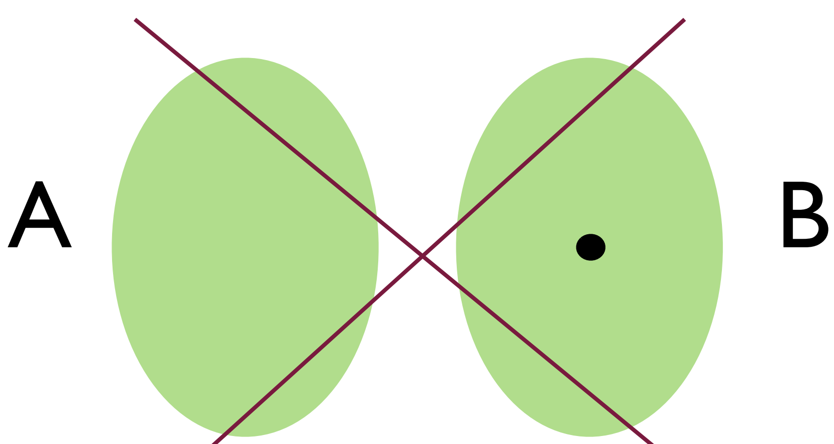


Special functions

The number of ingoing arrows for a function can be 0,1, or more.
Based on this, we distinguish some special functions.



at most one incoming arrow
injection



at least one incoming arrow
surjection

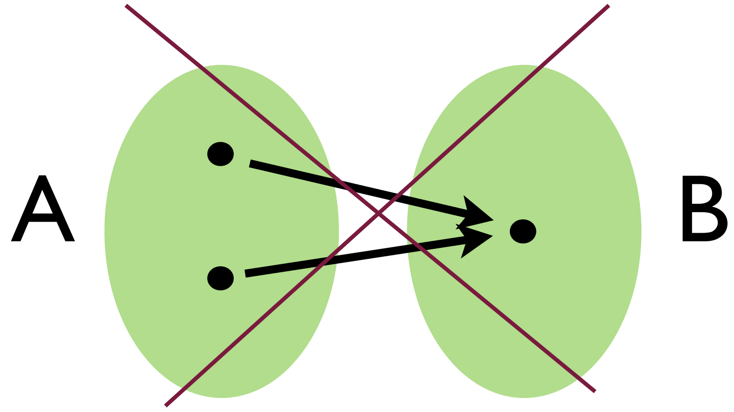
exactly one incoming arrow (injection + surjection) bijection

Special functions

Special functions

Def. A function $f: A \longrightarrow B$ is injective iff

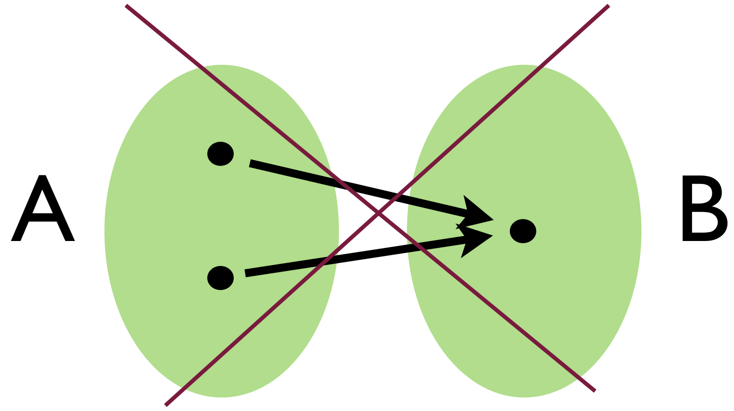
for all $a, b \in A$, if $f(a) = f(b)$ then $a = b$.



Special functions

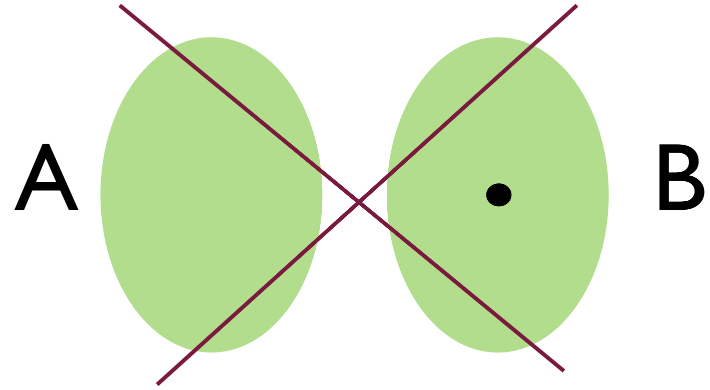
Def. A function f: A $\longrightarrow$ B is injective iff

for all a, b ∈ A, if f(a) = f(b) then a = b.



Def. A function f: A $\longrightarrow$ B is surjective iff

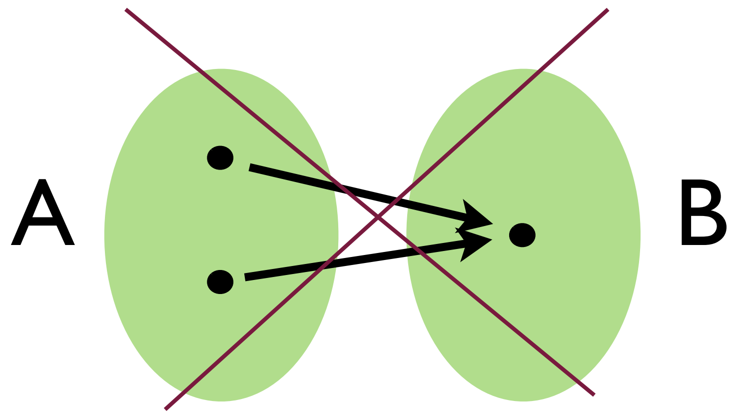
for all b ∈ B, there exists a ∈ A such that f(a) = b.



Special functions

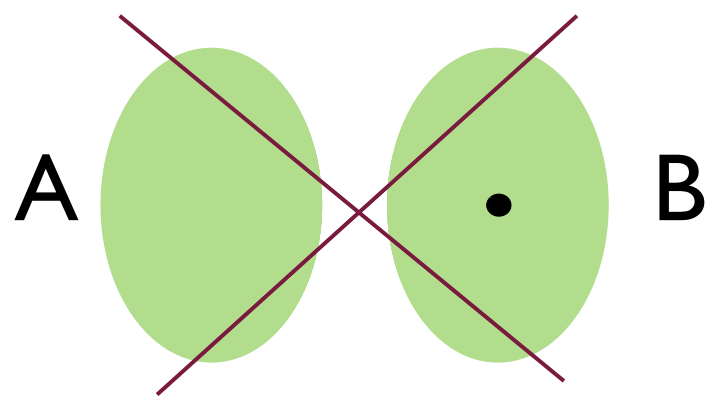
Def. A function $f: A \longrightarrow B$ is injective iff

for all $a, b \in A$, if $f(a) = f(b)$ then $a = b$.



Def. A function $f: A \longrightarrow B$ is surjective iff

for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.



Def. A function $f: A \longrightarrow B$ is bijective iff

f is injective and surjective.

Simple characterisations

Simple characterisations

Lemma 11: A function $f: A \longrightarrow B$ is injective iff

for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

Simple characterisations

Lemma 11: A function $f: A \longrightarrow B$ is injective iff

for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

at most one incoming arrow

injection

Simple characterisations

Lemma I1: A function $f: A \longrightarrow B$ is injective iff

for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

at most one incoming arrow
injection

Lemma S1: A function $f: A \longrightarrow B$ is surjective iff

$|f^{-1}(\{b\})| \geq 1$ for all $b \in B$ iff

$f(A) = B$.

Simple characterisations

Lemma I1: A function $f: A \longrightarrow B$ is injective iff

for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

at most one incoming arrow
injection

Lemma S1: A function $f: A \longrightarrow B$ is surjective iff

$|f^{-1}(\{b\})| \geq 1$ for all $b \in B$ iff

$f(A) = B$.

at least one incoming arrow
surjection

Simple characterisations

Lemma I1: A function $f: A \longrightarrow B$ is injective iff

for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

at most one incoming arrow
injection

Lemma S1: A function $f: A \longrightarrow B$ is surjective iff

$|f^{-1}(\{b\})| \geq 1$ for all $b \in B$ iff
$f(A) = B$.

at least one incoming arrow
surjection

Lemma B1: A function $f: A \longrightarrow B$ is bijective iff

$|f^{-1}(\{b\})| = 1$ for all $b \in B$ iff
f is both injective and surjective.

Simple characterisations

Lemma I1: A function $f:A \longrightarrow B$ is injective iff

for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

at most one incoming arrow
injection

Lemma S1: A function $f:A \longrightarrow B$ is surjective iff

$|f^{-1}(\{b\})| \geq 1$ for all $b \in B$ iff
$f(A) = B$.

at least one incoming arrow
surjection

Lemma B1: A function $f:A \longrightarrow B$ is bijective iff

$|f^{-1}(\{b\})| = 1$ for all $b \in B$ iff
f is both injective and surjective.

exactly one incoming arrow
bijection

Some properties

Some properties

Lemma 12: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f(x) \in f(A')$ iff $x \in A'$.

Some properties

Lemma 12: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f(x) \in f(A')$ iff $x \in A'$.

if holds always!

Some properties

Lemma 12: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f(x) \in f(A')$ iff $x \in A'$.

if holds always!

Prop. 13: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f^{-1}(f(A')) = A'$.

Some properties

Lemma I2: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f(x) \in f(A')$ iff $x \in A'$.

if holds always!

Prop. I3: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f^{-1}(f(A')) = A'$.

Prop. S2: Let $f: A \longrightarrow B$ be surjective and let $B' \subseteq B$. Then

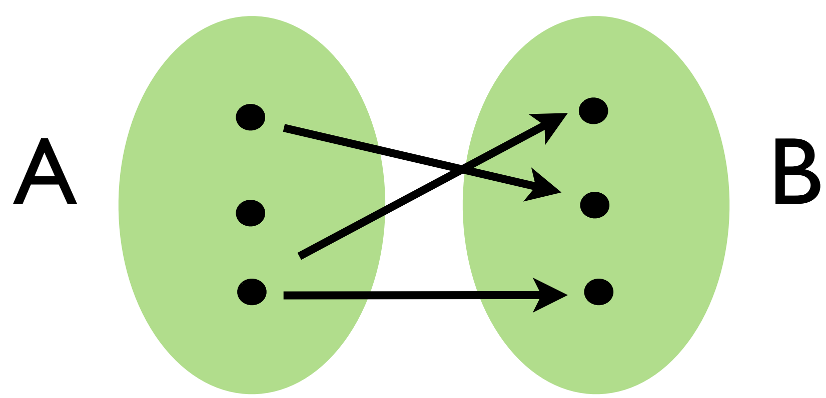
$f(f^{-1}(B')) = B'$.

Inverse function

Let $f: A \longrightarrow B$ be a **bijection**

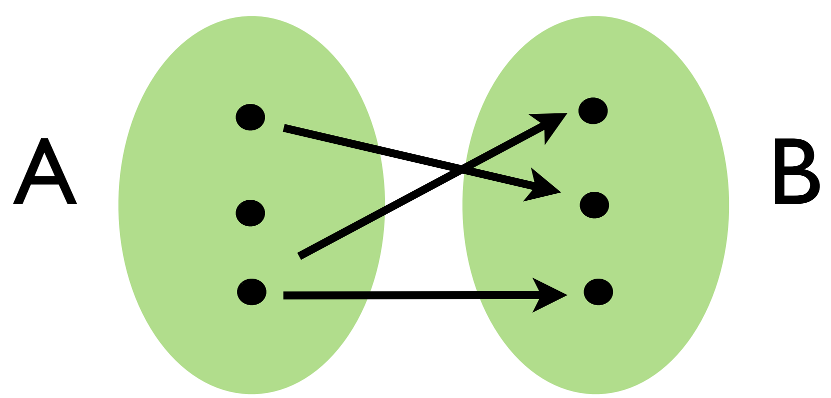
Inverse function

Let f:A $\longrightarrow$ B be a bijection



Inverse function

Let f:A ⟶ B be a bijection



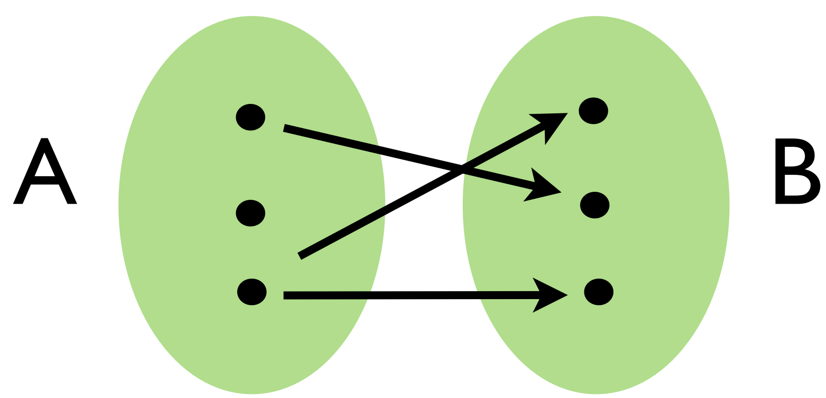
Def. The inverse function f^{-1}: B ⟶ A is defined as

$$f^{-1}(b) = a \text{ iff } f(a) = b, \quad b \in B.$$

Inverse function

Let $f{:}A \longrightarrow B$ be a bijection

well defined only if f is bijective!



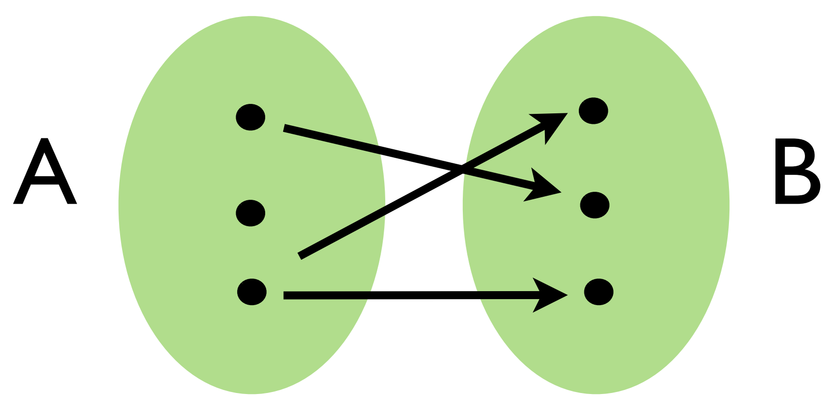
Def. The inverse function $f^{-1}: B \longrightarrow A$ is defined as

$$f^{-1}(b) = a \;\text{ iff }\; f(a) = b, \quad b \in B.$$

Inverse function

Let $f: A \longrightarrow B$ be a bijection

well defined only if f is bijective!



Def. The inverse function $f^{-1}: B \longrightarrow A$ is defined as

$$f^{-1}(b) = a \text{ iff } f(a) = b, \quad b \in B.$$

Lemma B2: The inverse function f^{-1} for a bijection f is bijective.

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

Def. The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by

$$g \circ f\,(a) = g(f(a)), \text{ for } a \in A.$$

Function composition

Let $f\!: A \longrightarrow B$ and $g\!: B \longrightarrow C$

"after"

$g \circ f : A \longrightarrow B \longrightarrow C$

Def. The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by

$$g \circ f\,(a) = g(f(a)), \text{ for } a \in A.$$

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

"after"

$g \circ f : A \longrightarrow B \longrightarrow C$

Def. The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by

$$g \circ f\,(a) = g(f(a)), \quad \text{for } a \in A.$$

Lemma 14: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be injective. Then $g \circ f$ is injective.

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

"after"

$g \circ f : A \longrightarrow B \longrightarrow C$

Def. The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by

$$g \circ f\,(a) = g(f(a)), \text{ for } a \in A.$$

Lemma I4: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be injective. Then $g \circ f$ is injective.

Lemma S3: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be surjective. Then $g \circ f$ is surjective.

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

"after"
$g \circ f : A \longrightarrow B \longrightarrow C$

Def. The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by

$$g \circ f\,(a) = g(f(a)), \text{ for } a \in A.$$

Lemma I4: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be injective. Then $g \circ f$ is injective.

Lemma S3: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be surjective. Then $g \circ f$ is surjective.

Corollary B2: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be bijective. Then so is $g \circ f$.

A characterization of bijections

A characterization of bijections

Theorem B3: A function $f: A \longrightarrow B$ is bijective iff there exists a function $g: B \longrightarrow A$ with

$$g \circ f = id_A \text{ and } f \circ g = id_B.$$

A characterization of bijections

Theorem B3: A function $f: A \longrightarrow B$ is bijective iff there exists a function $g: B \longrightarrow A$ with $g \circ f = id_A$ and $f \circ g = id_B$.

$id_A: A \longrightarrow A$,
$id_A(a) = a$, for all $a \in A$