Cardinality

Cardinals

Def. Two sets A and B have the same cardinality (are equinumerous) if there is a bijection $f: A \to B$. Notation $A \sim B$, or $|A| = |B|$.

Prop. The relation $\sim$ is an equivalence relation on sets.

$|A| = [A]_\sim$

cardinal numbers are $\sim$ equivalence classes

Def. A set A has at most as large cardinality as a set B if there is an injection $f: A \to B$. Notation $|A| \leq |B|$.

Def. A set A has at least as large cardinality as a set B if there is a surjection $f: A \to B$. Notation $|A| \geq |B|$.

Def. A set A has smaller cardinality than a set B if there is an injection $f: A \to B$ and there is no surjection $f: A \to B$. Notation $|A| < |B|$.

Theorem (Cantor)

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Operations on cardinals

$|A| = [A]_\sim$

cardinal numbers are ~ equivalence classes

Def. Let A and B be two disjoint sets. Then $|A| + |B| = |A \cup B|$.

Def. Let A and B be two sets. Then $|A| \cdot |B| = |A \times B|$.

Def. Let A and B be two sets. Then $|A|^{|B|} = |A^B|$ where A^B is the set of all functions from B to A, i.e. $A^B = \{f \mid f: B \to A\}$.

Prop. Let A be a set. Then $|\mathcal{P}(A)| = 2^{|A|}$.

Note: $2 = |\{0,1\}|$

Finite sets, finite cardinals

We write $\mathbb{N}_k$ for the set $\{0,1,..., k-1\}$. Then $\mathbb{N}_0 = \varnothing$.

We will also write k for $|\mathbb{N}_k|$.

$|A| = [A]_\sim$

cardinal numbers are ~ equivalence classes

Def. A set A is finite if and only if $|A| = |\mathbb{N}_k|$, for some $k \in \mathbb{N}$. We write then $|A| = k$.

Hence A set A is finite if and only if there is a natural number $k \in \mathbb{N}$ and a bijection $f: A \rightarrow \mathbb{N}_k$.

if and only if A has k elements, for some $k \in \mathbb{N}$

E.g. If $|A| = k$ and $|B| = m$ for some $k,m \in \mathbb{N}$ then $|A \times B| = k \cdot m$

The operations on cardinals when restricted to finite cardinals coincide with the operations on natural numbers! This justifies the notation.

Infinite, countable and uncountable sets

Time for a video!

Hilbert’s
infinite hotel :-)

Infinite, countable and uncountable sets

We write $\aleph_0$ or the cardinality of natural numbers.
Hence $\aleph_0 = |\mathbb{N}|$.

$|A| = [A]_\sim$

cardinal numbers are $\sim$ equivalence classes

Def. A set A is countable iff $|A| = \aleph_0$.

Prop.
$\mathbb{N}$ is countable.
$\mathbb{Z}$ is countable.
$\mathbb{Q}$ is countable.

Def. A set is infinite iff $|A| \geq \aleph_0$.

Hence, every countable set is infinite

Def. A set is uncountable iff $|A| > \aleph_0$.

Prop. $\mathbb{R}$ is uncountable.

We write c for $|\mathbb{R}|$

Cardinals are unbounded

$|A| = [A]_{\sim}$

cardinal numbers are ~ equivalence classes

Theorem (Cantor)

For every set A we have $|A| < |\mathcal{P}(A)|$.

Hence, for every cardinal there is a larger one.

Finite Automata

Alphabets and Languages

Def

$\Sigma^0 = \{\mathcal{E}\}$ contains only the empty word

Σ - alphabet (finite set)

$\Sigma^n = \{a_1 a_2 .. a_n \mid a_i \in \Sigma\}$ is the set of words of length n

$\Sigma^* = \{w \mid \exists n \in \mathbb{N}.\ \exists\ a_1, a_2, .., a_n \in \Sigma.\ w = a_1 a_2 .. a_n\}$ is the set of all words over Σ

A language L over Σ is a subset $L \subseteq \Sigma^*$

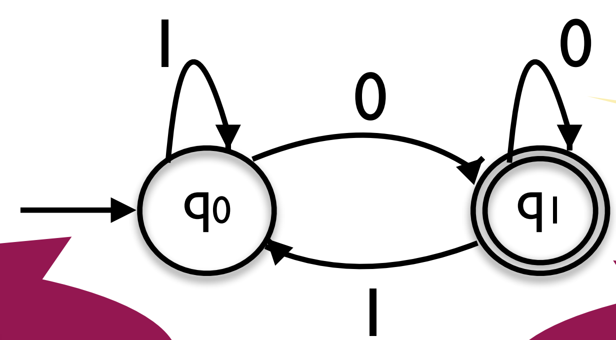
Deterministic Automata (DFA)

Informal example

alphabet

$\Sigma = \{0,1\}$

M_1:



q_0, q_1 are states

transitions, labelled by alphabet symbols

q_0 is initial

q_1 is final

Accepts the language $L(M_1) = \{w \in \Sigma^* \mid w \text{ ends with a } 0\} = \Sigma^*0$

regular language

regular expression

DFA

Definition

A deterministic automaton M is a tuple $M = (Q, \Sigma, \delta, q_0, F)$ where

Q is a finite set of states
Σ is a finite alphabet
$\delta: Q \times \Sigma \longrightarrow Q$ is the transition function
q_0 is the initial state, $q_0 \in Q$
F is a set of final states, $F \subseteq Q$

In the example M_1

$M_1 = (Q, \Sigma, \delta, q_0, F)$ for

$Q = \{q_0, q_1\}$ $F = \{q_1\}$

$\Sigma = \{0, 1\}$

$\delta(q_0, 0) = q_1$, $\delta(q_0, 1) = q_0$
$\delta(q_1, 0) = q_1$, $\delta(q_1, 1) = q_0$

DFA

The extended transition function

Given $M = (Q, \Sigma, \delta, q_0, F)$ we can extend $\delta: Q \times \Sigma \longrightarrow Q$ to

$$\delta^*: Q \times \Sigma^* \longrightarrow Q$$

In M_1, $\delta^*(q_0, 110010) = q_1$

inductively, by:

$$\delta^*(q, \varepsilon) = q \quad \text{and} \quad \delta^*(q, wa) = \delta(\delta^*(q, w), a)$$

Definition

The language recognised / accepted by a deterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ is

$$L(M) = \{w \in \Sigma^* \mid \delta^*(q_0, w) \in F\}$$

$L(M_1) = \{w0 \mid w \in \{0,1\}^*\}$