Transitive closure

Let R be a relation on a set X. The transitive closure (transitive Hülle) of R, notation R^+, is the relation

$$R^+ = \bigcup_{n \in \mathbb{N}, n \neq 0} R^n$$

$R^{n+1} = R^n \circ R$

The reflexive and transitive closure (reflexive und transitive Hülle) of R, notation R^*, is the relation

$$R^* = \bigcup_{n \in \mathbb{N}} R^n$$

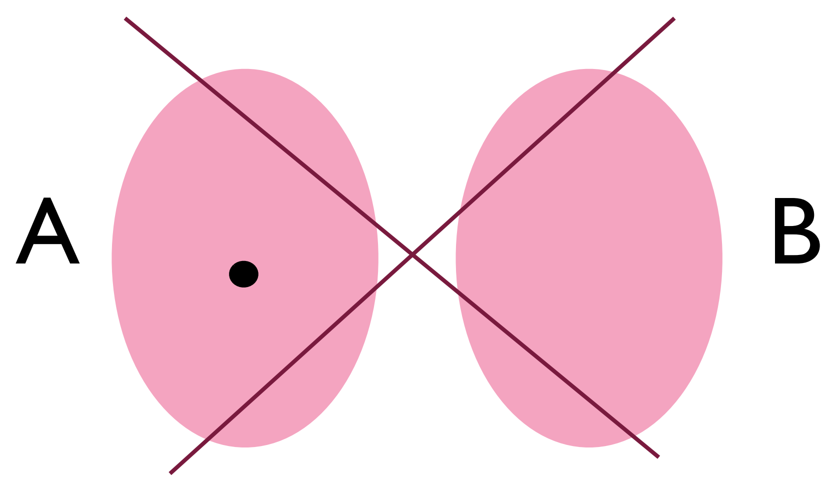
$R^0 = \Delta_X$

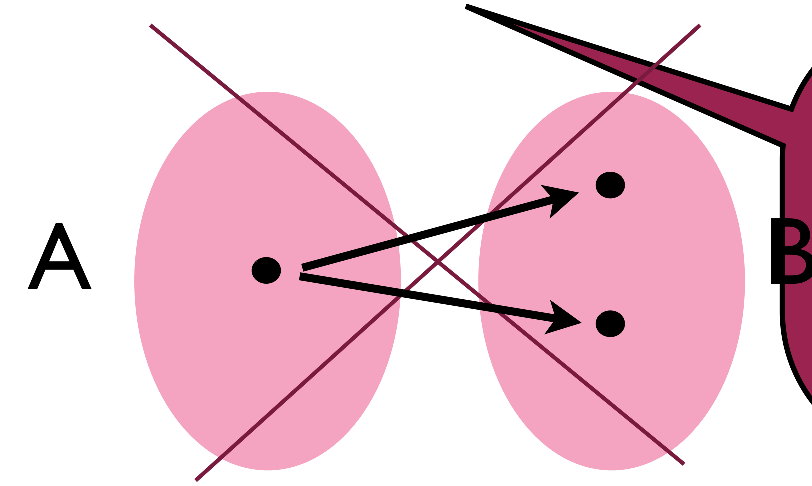
Proposition TC: Let R be a relation on X. The transitive closure of R is the smallest transitive relation that contains R. The reflexive and transitive closure of R is the smallest reflexive and transitive relation that contains R.

Functions, mappings

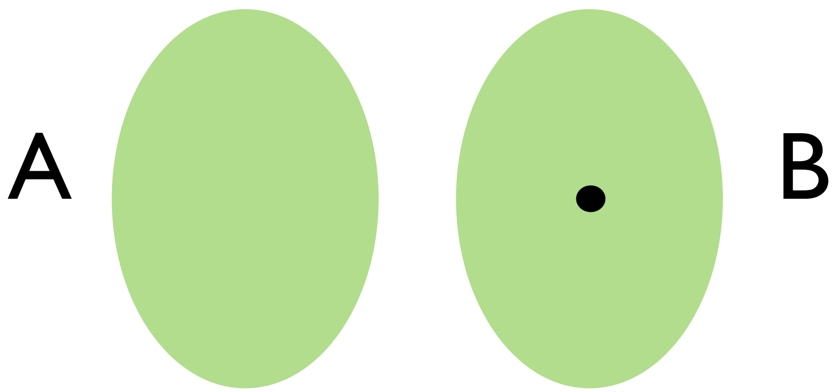
Def. If A and B are sets, then F is a function (mapping, Abbildung) from A to B, notation F: A $\longrightarrow$ B iff

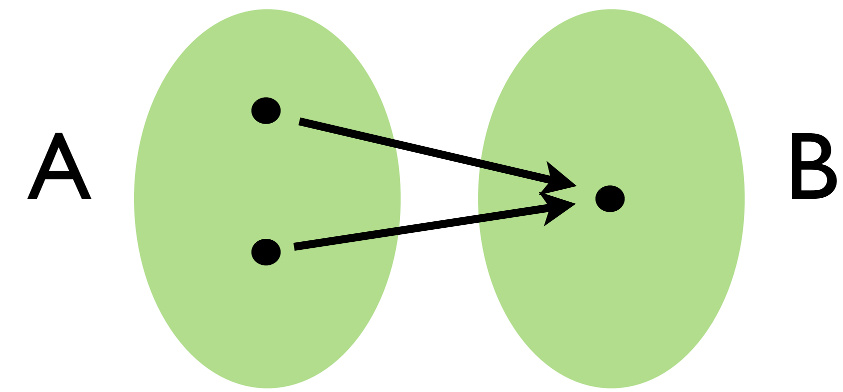
for every a $\in$ A, there exists a unique b $\in$ B such that b = F(a).





the image of a,
b = F(a)
a $\mapsto$ b





{(a, F(a)) | a $\in$ A} is the graph of the function F

Functions, mappings

When $f\colon A \longrightarrow B$ then dom $f = A$ and cod $f = B$

domain of F
(Definitionsbereich)

codomain of F
(Wertebereich)

Let $f\colon A \longrightarrow B$ and $A' \subseteq A$.

The image (Bild) of A' is the set $f(A') = \{f(a) \mid a \in A'\} \subseteq B$.

$f(A') = \{b \in B \mid \text{there is an } a \in A' \text{ with } b = f(a)\}$

if $a \in A'$, then $f(a) \in f(A')$

So f extends to a function $f\colon \mathcal{P}(A) \longrightarrow \mathcal{P}(B)$, the image-function.

Functions, mappings

Let f: $A \longrightarrow B$ and $B' \subseteq B$.

The inverse image (Urbild) of B' is the set

$$f^{-1}(B') = \{a \mid f(a) \in B'\} \subseteq A.$$

$a \in f^{-1}(B')$ iff $f(a) \in B'$

Again the inverse image induces a function $f^{-1}: \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$, the inverse-image-function.

Lemma F1: Let f: $A \longrightarrow B$, $A' \subseteq A$, and $B' \subseteq B$. Then

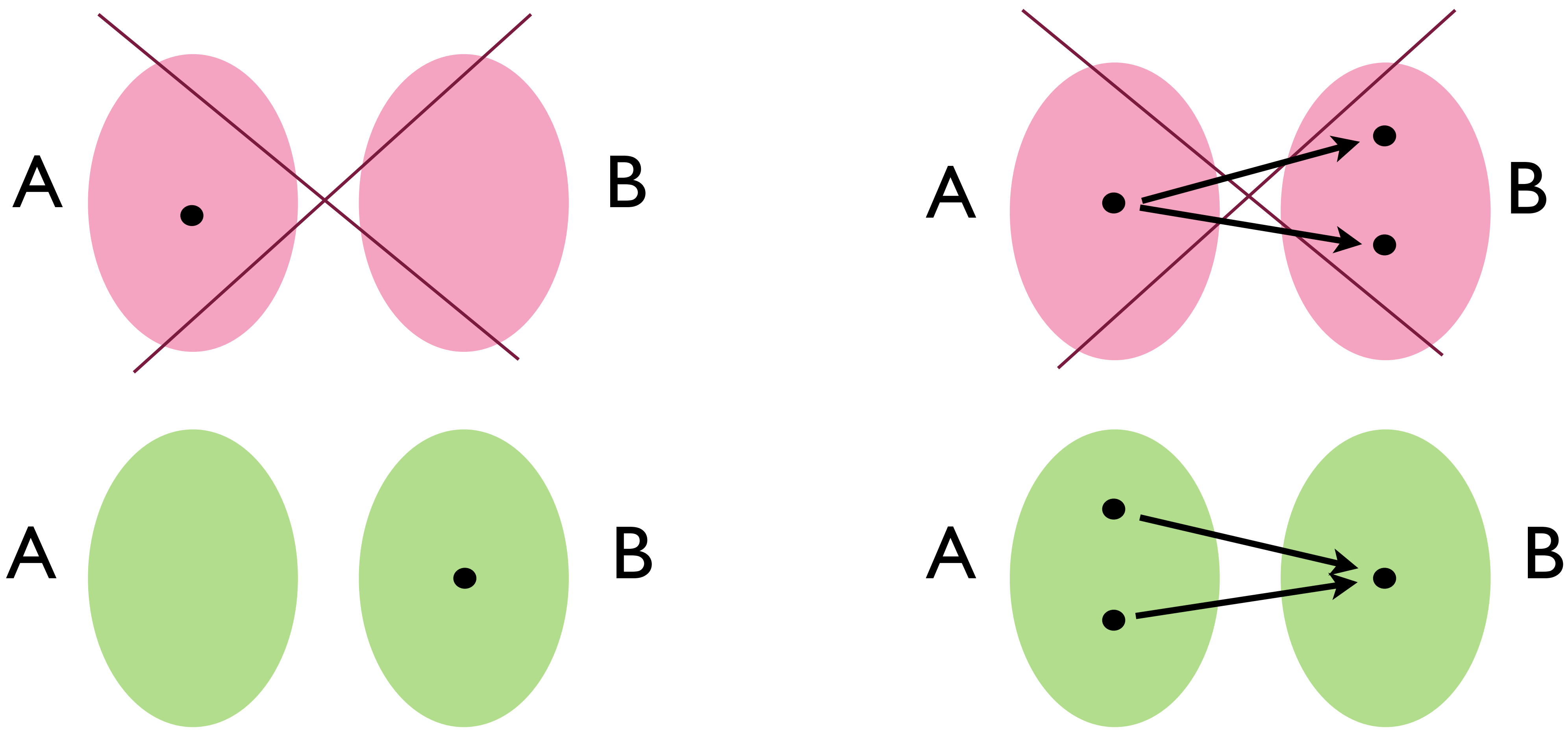
$$A' \subseteq f^{-1}(f(A')) \quad \text{and} \quad f(f^{-1}(B')) \subseteq B'$$

(in general no more than this holds)

Recall…

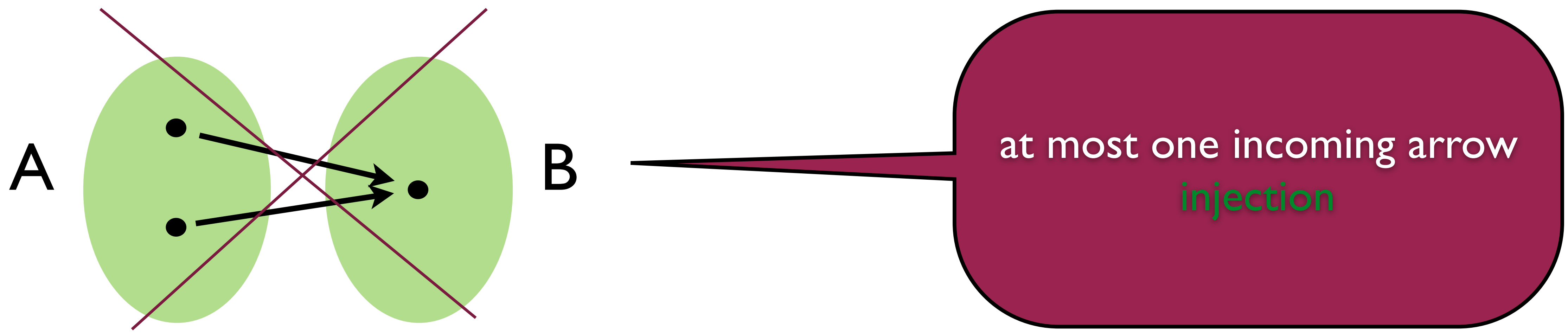
Def. If A and B are sets, then F is a function (mapping, Abbildung) from A to B, notation $F: A \longrightarrow B$ iff

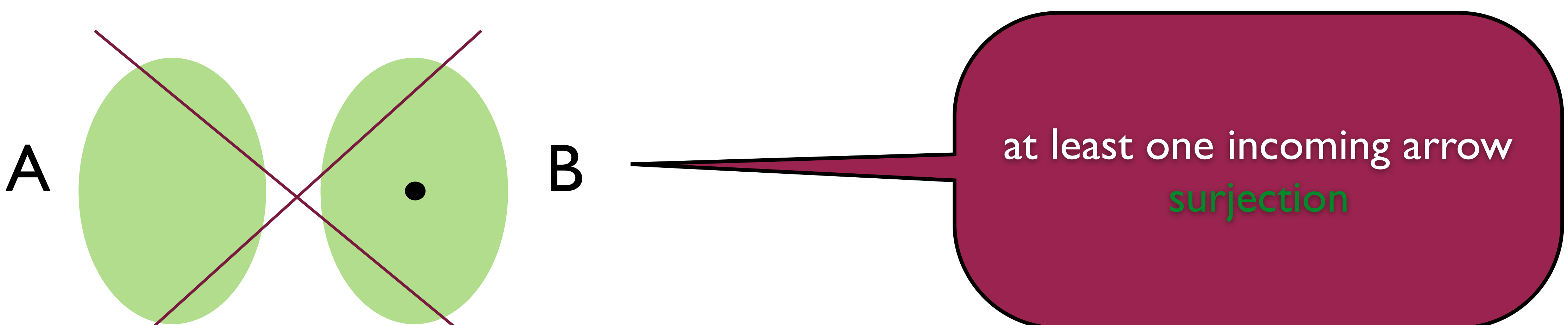
for every $a \in A$, there exists a unique $b \in B$ such that $b = F(a)$.



Special functions

The number of ingoing arrows for a function can be 0, 1, or more. Based on this, we distinguish some special functions.



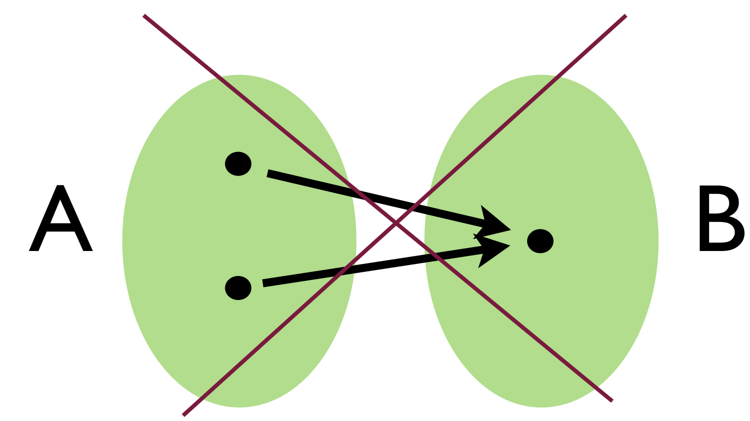


exactly one incoming arrow (injection + surjection) bijection

Special functions

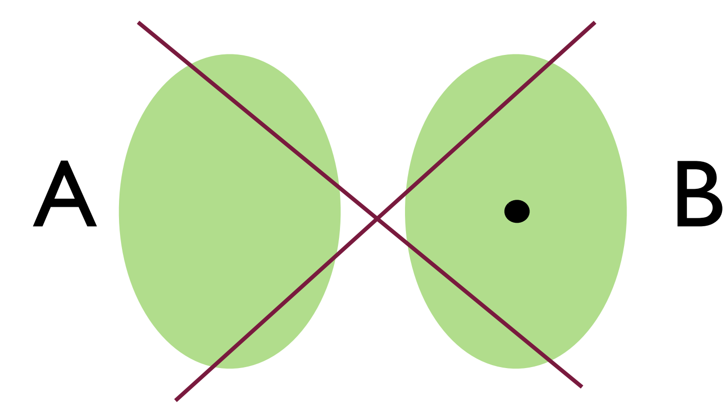
Def. A function $f: A \longrightarrow B$ is injective iff

for all $a, b \in A$, if $f(a) = f(b)$ then $a = b$.



Def. A function $f: A \longrightarrow B$ is surjective iff

for all $b \in B$, there exists $a \in A$ such that $f(a) = b$.



Def. A function $f: A \longrightarrow B$ is bijective iff

for all $b \in B$, there exists unique $a \in A$ with $f(a) = b$.

Simple characterisations

Lemma I1: A function $f: A \longrightarrow B$ is injective iff

for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

at most one incoming arrow
injection

Lemma S1: A function $f: A \longrightarrow B$ is surjective iff

$|f^{-1}(\{b\})| \geq 1$ for all $b \in B$ iff
$f(A) = B$.

at least one incoming arrow
surjection

Lemma B1: A function $f: A \longrightarrow B$ is bijective iff

$|f^{-1}(\{b\})| = 1$ for all $b \in B$ iff
f is both injective and surjective.

exactly one incoming arrow
bijection

Some properties

Lemma I2: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f(x) \in f(A')$ iff $x \in A'$.

if holds always!

Prop. I3: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f^{-1}(f(A')) = A'$.

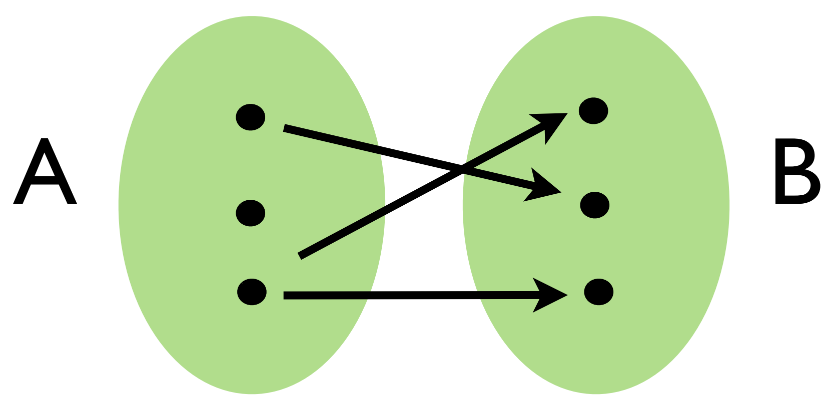
Prop. S2: Let $f: A \longrightarrow B$ be surjective and let $B' \subseteq B$. Then

$f(f^{-1}(B')) = B'$.

Inverse function

Let $f{:}A \longrightarrow B$ be a bijection

well defined only if f is bijective!



Def. The inverse function $f^{-1}: B \longrightarrow A$ is defined as

$$f^{-1}(b) = a \quad \text{iff} \quad f(a) = b, \quad b \in B.$$

Lemma B2: The inverse function f^{-1} for a bijection f is bijective.

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

Function composition

Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$

"after"

$g \circ f : A \longrightarrow B \longrightarrow C$

Def. The composition $g \circ f$ is a function $g \circ f : A \longrightarrow C$ given by

$$g \circ f\,(a) = g(f(a)), \text{ for } a \in A.$$

Lemma I4: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be injective. Then $g \circ f$ is injective.

Lemma S3: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be surjective. Then $g \circ f$ is surjective.

Corollary B2: Let $f: A \longrightarrow B$ and $g: B \longrightarrow C$ be bijective. Then so is $g \circ f$.

A characterization of bijections

Theorem B3: A function $f: A \longrightarrow B$ is bijective iff

there exists a function $g: B \longrightarrow A$ with

$g \circ f = id_A$ and $f \circ g = id_B$.

$id_A: A \longrightarrow A$,

$id_A(a) = a$, for all $a \in A$

Equality of functions

Let $f: A \longrightarrow B$ and $g: C \longrightarrow D$

Def. The functions $f: A \longrightarrow B$ and $g: C \longrightarrow D$ are equal iff

(1) $A = C$

(2) $B = D$

(3) for all $a \in A$, $f(a) = g(a)$.

dom f = dom g

cod f = cod g