Relations

Def. If A and B are sets, then any subset $R \subseteq A \times B$ is a (binary) relation between A and B

similarly, unary relation (subset), n-ary relation...

Def. R is a relation on A if $R \subseteq A \times A$

some relations are special

Operations on relations

Let $R \subseteq A \times \underline{B}$ and $S \subseteq \underline{B} \times C$ be two relations. Their composition is the relation

$$R \circ S = \{(a,c) \in A \times C \mid \text{there is } b \in B \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\}$$

relational composition is associative $(R \circ S) \circ T = R \circ (S \circ T)$

so again we write

$$R^n = \underbrace{R \circ R \circ \ldots \circ R}_{n \text{ times}}$$

Let $R \subseteq A \times B$ be a relation. The inverse relation of R is the relation

$$R^{-1} = \{(b,a) \in B \times A \mid (a, b) \in R\}$$

Characterizations

Lemma: Let R be a relation over the set A. Then

1. R is reflexive iff $\Delta_A \subseteq R$
2. R is symmetric iff $R \subseteq R^{-1}$
3. R is transitive iff $R^2 \subseteq R$

Important equivalence on $\mathbb{Z}$

Def. For a natural number n, the relation $\equiv_n$ is defined as

$i \equiv_n j$ iff $n \mid i - j$

[iff i-j is a multiple of n]

[iff there exists $k \in \mathbb{Z}$ s.t. $i-j = k \cdot n$]

[iff $\exists k \, (k \in \mathbb{Z} \wedge i-j = k \cdot n)$]

logical formula

Lemma: The relation $\equiv_n$ is an equivalence for every n.

Equivalences classes

Def. Let R be an equivalence over A and $a \in A$. Then

$$[a]_R = \{ b \in A \mid (a, b) \in R\}$$

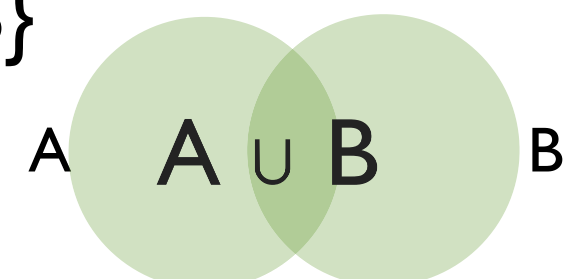
the equivalence class of a

Lemma E1: Let R be an equivalence over the set A. Then
for all $a, b \in A$, $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \varnothing$

Task: Describe the equivalence classes of $\equiv_n$
How many classes are there?

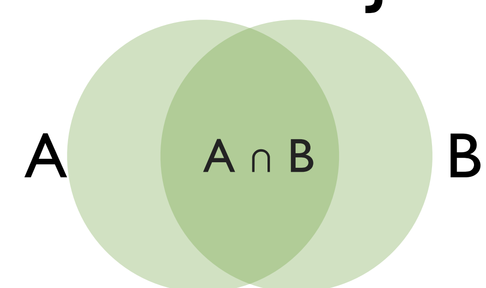
Unions and intersections of multiple sets

Union (Vereinigung) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



Intersection (Durchschnitt) $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

A and B are disjoint if $A \cap B = \emptyset$



In general, for sets $A_1, A_2, \ldots, A_n$ with $n \geq 1$,

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{1 \leq i \leq n} A_i = \{x \mid x \in A_i \text{ for some } i \in \{1,..n\}\}$$

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{1 \leq i \leq n} A_i = \{x \mid x \in A_i \text{ for all } i \in \{1,..n\}\}$$

Unions and intersections of multiple sets

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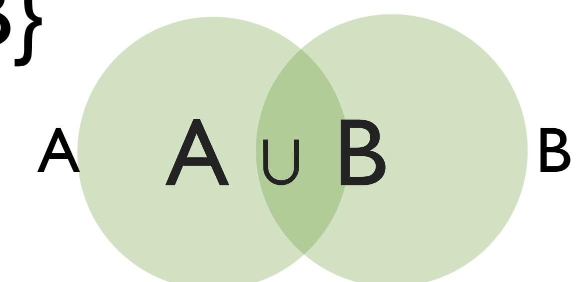

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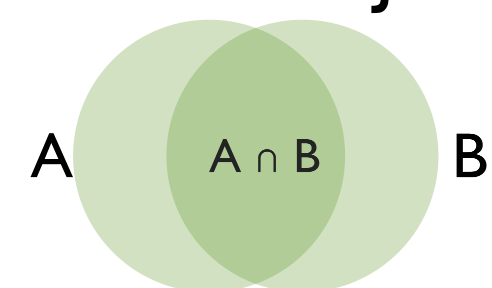


In general, for a family of sets $(A_i \mid i \in I)$

$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

Back to equivalence classes

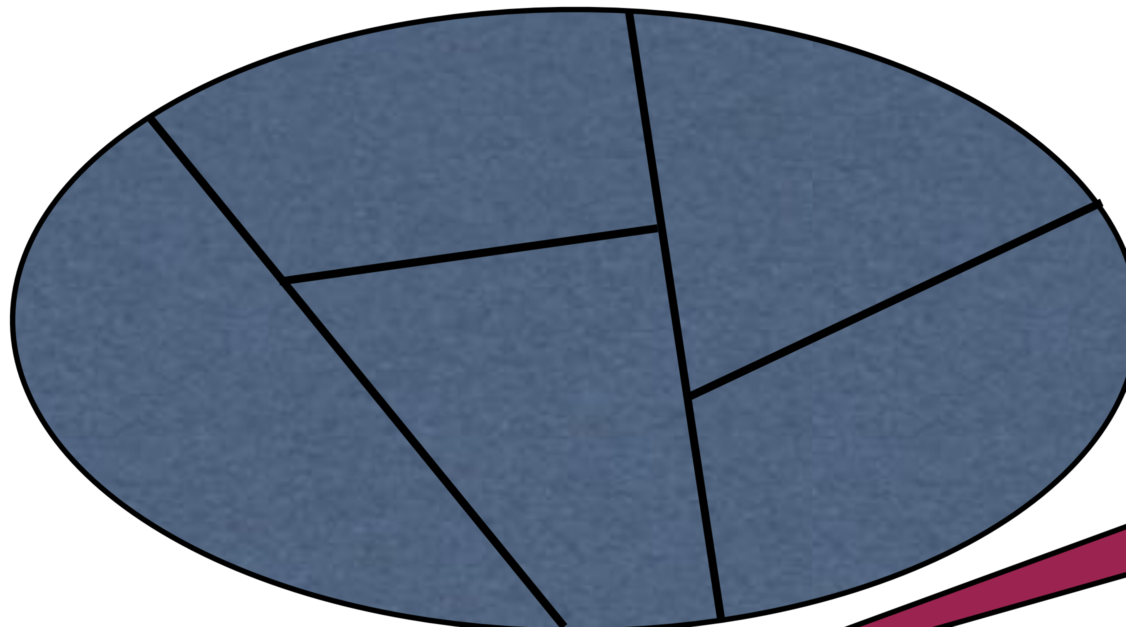
Example: Let R be an equivalence over A and $a \in A$. Then

$$([a]_R , a \in A)$$ is a family of sets.

all equivalence classes of R

Lemma E2: $A = \bigcup_{a \in A} [a]_R$. The union is disjoint.

Partitions



hence, a collection of subsets of X

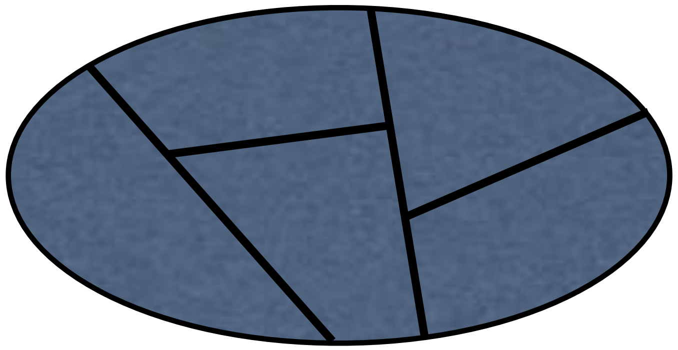
Def. Let X be a set. A subset P of the powerset $\mathcal{P}(X)$ is a partition (Klasseneinteilung) of X if it satisfies:

(1) For all $A \in P$, $A \neq \emptyset$

(2) For all $A, B \in P$, if $A \neq B$ then $A \cap B = \emptyset$

(3) $\bigcup_{A \in P} A = X$

that are non-empty, pairwise disjoint, and their union equals X



Partitions = Equivalences

Theorem PE: Let X be a set.

(1) If R is an equivalence on X, then the set

$$P(R) = \{ [x]_R \mid x \in X \}$$

is a partition of X.

(2) If P is a partition of X, then the relation

$$R(P) = \{(x,y) \in X \times X \mid \text{there is } A \in P \text{ such that } x,y \in A\}$$

is an equivalence relation.

Moreover, the assignments $R \mapsto P(R)$ and $P \mapsto R(P)$ are inverse to each other, i.e., $R(P(R)) = R$ and $P(R(P)) = P$.