Product of multiple sets

Direct product (Kartesisches Produkt)

$$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$$

ordered pairs

$(A \times B) \times C \neq A \times (B \times C)$

Therefore, we define

$A \times B \times C =$ [illegible] and $y \in B$ and $z \in C\}$

if $A_i = A$ for all i, then the product is denoted A^n

In general, for sets $A_1, A_2, \ldots, A_n$ with $n \geq 1$,

sequence of length n

$$A_1 \times A_2 \times \ldots \times A_n = \prod_{1 \leq i \leq n} A_i = \{(x_1,x_2,\ldots,x_n) \mid x_i \in A_i \text{ for } 1 \leq i \leq n\}$$

Relations

Def. If A and B are sets, then any subset $R \subseteq A \times B$ is a (binary) relation between A and B

similarly, unary relation (subset), n-ary relation...

Def. R is a relation on A if $R \subseteq A \times A$

some relations are special

Special relations

A relation $R \subseteq A \times A$ is:

reflexive	iff	for all $a \in A$, $(a,a) \in R$
symmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \in R$
transitive	iff	for all $a,b,c \in A$, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$
irreflexive	iff	for all $a \in A$, $(a,a) \notin R$
antisymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$
asymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \notin R$
total	iff	for all $a,b \in A$, $(a,b) \in R$ or $(b,a) \in R$

(infix) notation aRb for $(a,b) \in R$

Special relations

A relation R on A, i.e., $R \subseteq A \times A$ is:

equivalence	iff	R is reflexive, symmetric, and transitive
partial order	iff	R is reflexive, antisymmetric, and transitive
strict order	iff	R is irreflexive and transitive
preorder	iff	R is reflexive and transitive
total (linear) order	iff	R is a total partial order

Obvious properties

1. Every partial order is a preorder.

2. Every total order is a partial order.

3. Every total order is a preorder.

4. If $R \subseteq A \times A$ is a relation such that there are $a, b \in A$ with

$$a \neq b, (a,b) \in R \text{ and } (b,a) \in R,$$

then R is not a partial order, nor a total order, nor a strict order.

Operations on relations

Let $R \subseteq A \times \underline{B}$ and $S \subseteq \underline{B} \times C$ be two relations. Their composition is the relation

$$R \circ S = \{(a,c) \in A \times C \mid \text{there is } b \in B \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\}$$

relational composition is associative $(R \circ S) \circ T = R \circ (S \circ T)$

so again we write
$R^n = \underbrace{R \circ R \circ \ldots \circ R}_{n \text{ times}}$

Let $R \subseteq A \times B$ be a relation. The inverse relation of R is the relation

$$R^{-1} = \{(b,a) \in B \times A \mid (a, b) \in R\}$$

Characterizations

Lemma: Let R be a relation over the set A. Then

1. R is reflexive iff $\Delta_A \subseteq R$
2. R is symmetric iff $R \subseteq R^{-1}$
3. R is transitive iff $R^2 \subseteq R$