Equivalences with quantifiers

Renaming bound variables

Bound variables

$$\forall_x [P{:}Q] \stackrel{val}{=} \forall_y [P[y/x]{:}Q[y/x]]$$

$$\exists_x [P{:}Q] \stackrel{val}{=} \exists_y [P[y/x]{:}Q[y/x]]$$

if y does not occur in P or Q (not even in ∀y, ∃y)

Domain splitting

Examples:

$$\forall_x[x \leqslant 1 \vee x \geqslant 5 \colon x^2 - 6x + 5 \geqslant 0]$$
$$\overset{val}{=} \forall_x[x \leqslant 1 \colon x^2 - 6x + 5 \geqslant 0] \wedge \forall_x[x \geqslant 5 \colon x^2 - 6x + 5 \geqslant 0]$$

$$\exists_k[0 \leqslant k \leqslant n : k^2 \leqslant 10]$$
$$\overset{val}{=} \exists_k[0 \leqslant k \leqslant n-1 \vee k = n : k^2 \leqslant 10]$$
$$\overset{val}{=} \exists_k[0 \leqslant k \leqslant n-1 : k^2 \leqslant 10] \vee \exists_k[k = n : k^2 \leqslant 10]$$

Domain splitting

$$\forall_x[P \vee Q : R] \overset{val}{=} \forall_x[P : R] \wedge \forall_x[Q : R]$$
$$\exists_x[P \vee Q : R] \overset{val}{=} \exists_x[P : R] \vee \exists_x[Q : R]$$

Equivalences with quantifiers

One-element domain

$$\forall_x[x = n : Q] \overset{val}{=} Q[n/x]$$

$$\exists_x[x = n : Q] \overset{val}{=} Q[n/x]$$

Example:

$$\forall_x[x = 3 : 2 \cdot x \geqslant 1] \overset{val}{=} 2 \cdot 3 \geqslant 1$$

Empty domain

$$\forall_x[F : Q] \overset{val}{=} T$$

$$\exists_x[F : Q] \overset{val}{=} F$$

"All Marsians are green"

Domain weakening

Intuition: The following are equivalent

$$\forall_x[x \in D : A(x)] \quad \text{and} \quad \forall_x[x \in D \Rightarrow A(x)]$$
$$\exists_x[x \in D : A(x)] \quad \text{and} \quad \exists_x[x \in D \wedge A(x)]$$

The same can be done to parts of the domain

Domain weakening

$$\forall_x[P \wedge Q : R] \overset{val}{=} \forall_x[P : Q \Rightarrow R]$$
$$\exists_x[P \wedge Q : R] \overset{val}{=} \exists_x[P : Q \wedge R]$$

$$P \wedge Q \overset{val}{\models} P$$

De Morgan with quantifiers

De Morgan

$$\neg\forall_x[P:Q] \stackrel{val}{=} \exists_x[P:\neg Q]$$
$$\neg\exists_x[P:Q] \stackrel{val}{=} \forall_x[P:\neg Q]$$

not for all = at least for one not

not exists = for all not

Hence: $\neg\forall = \exists\neg$ and $\neg\exists = \forall\neg$

It holds further that:

$$\neg\forall_x\neg = \exists_x\neg\neg = \exists_x$$
$$\neg\exists_x\neg = \forall_x\neg\neg = \forall_x$$

Substitution

holds also for quantified formulas!

meta rule

Simple

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P] \overset{val}{=} \psi[\xi/P]}$$

EVERY occurrence of P is substituted!

Sequential

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{=} \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{=} \psi[\xi/P, \eta/Q]}$$

The rule of Leibniz

holds also for quantified formulas!

meta rule

Leibniz

$$\frac{\phi \overset{val}{=} \psi}{C[\phi] \overset{val}{=} C[\psi]}$$

formula that has ϕ as a sub formula

single occurrence is replaced!

Other equivalences with quantifiers

Exchange trick

$$\forall_x[P{:}Q] \overset{val}{=} \forall_x[\neg Q{:}\neg P]$$

$$\exists_x[P{:}Q] \overset{val}{=} \exists_x[Q{:}P]$$

No wonder as

$$\forall_x[P{:}Q] \overset{val}{=} \forall_x[P \Rightarrow Q]$$

$$\exists_x[P{:}Q] \overset{val}{=} \exists_x[P \wedge Q]$$

Term splitting

$$\forall_x[P{:}Q \wedge R] \overset{val}{=} \forall_x[P{:}Q] \wedge \forall_x[P{:}R]$$

$$\exists_x[P{:}Q \vee R] \overset{val}{=} \exists_x[P{:}Q] \vee \exists_x[P{:}R]$$

Other equivalences with quantifiers

Monotonicity of quantifiers

$$\forall_x[P{:}Q \Rightarrow R] \Rightarrow (\forall_x[P{:}Q] \Rightarrow \forall_x[P{:}R]) \stackrel{val}{=} T$$

$$\forall_x[P{:}Q \Rightarrow R] \Rightarrow (\exists_x[P{:}Q] \Rightarrow \exists_x[P{:}R]) \stackrel{val}{=} T$$

tautologies

Lemma E1: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Lemma W4: $P \stackrel{val}{\models} Q$ iff $P \Rightarrow Q$ is a tautology.

Lemma W5: If $Q \stackrel{val}{\models} R$ then $\forall_x[P{:}Q] \stackrel{val}{\models} \forall_x[P{:}R]$.

still hold (in predicate logic)