Substitution

meta rule

Simple

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P] \overset{val}{=} \psi[\xi/P]}$$

Sequential

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{=} \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{=} \psi[\xi/P, \eta/Q]}$$

EVERY occurrence of P is substituted!

The rule of Leibniz

meta rule

Leibniz

$$\frac{\phi \overset{val}{=} \psi}{C[\phi] \overset{val}{=} C[\psi]}$$

formula that has ϕ as a sub formula

single occurrence is replaced!

Strengthening and weakening

We had

Definition: Two abstract propositions P and Q are equivalent,
notation P $\stackrel{val}{=}$ Q, iff
(1) Always when P has truth value 1,
also Q has truth value 1, and
(2) Always when Q has truth value 1,
also P has truth value 1.

if we relax this, we get strengthening

Strengthening

Definition: The abstract proposition P is stronger than Q, notation $P \overset{val}{\vDash} Q$, iff

~~(1)~~ Always when P has truth value 1, also Q has truth value 1, ~~and~~
~~(2) Always when Q has truth value 1, also P has truth value 1.~~

Q is weaker than P

Strengthening

Definition: The abstract proposition P is stronger than Q, notation $P \vDash^{val} Q$, iff
  always when P has truth value 1,
  also Q has truth value 1.

always when P is true,
Q is also true

Q is weaker
than P

Properties

Lemma E1: $P \stackrel{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Lemma EW1: $P \stackrel{val}{=} Q$ iff $P \stackrel{val}{\models} Q$ and $Q \stackrel{val}{\models} P$.

Lemma W2: $P \stackrel{val}{\models} P$

Lemma W3:   If $P \stackrel{val}{\models} Q$ and $Q \stackrel{val}{\models} R$ then $P \stackrel{val}{\models} R$

Lemma W4: $P \stackrel{val}{\models} Q$ iff $P \Rightarrow Q$ is a tautology.

Standard Weakenings

and-or-weakening

$$P \wedge Q \overset{val}{\models} P$$

$$P \overset{val}{\models} P \vee Q$$

Extremes

$$\mathrm{F} \overset{val}{\models} P$$

$$P \overset{val}{\models} \mathrm{T}$$

Calculating with weakenings
(the use of standard weakenings)

Substitution

just holds

Simple

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P] \overset{val}{\models} \psi[\xi/P]}$$

Sequential

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{\models} \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{\models} \psi[\xi/P, \eta/Q]}$$

EVERY occurrence of P is substituted!

The rule of Leibniz

The rule of Leibniz

Leibniz

$$\frac{\phi \stackrel{val}{\models} \psi}{C[\phi] \stackrel{val}{\models} C[\psi]}$$

does not hold for weakening!

Monotonicity

$$\frac{P \stackrel{val}{\models} Q}{P \wedge R \stackrel{val}{\models} Q \wedge R}$$

$$\frac{P \stackrel{val}{\models} Q}{P \vee R \stackrel{val}{\models} Q \vee R}$$

Predicate logic

Limitations of propositional logic

Propositional logic only allows us to reason about completed statements about things, not about the things themselves.

Example

> Some chicken cannot fly
> All chicken are birds
> _____
> Some birds cannot fly

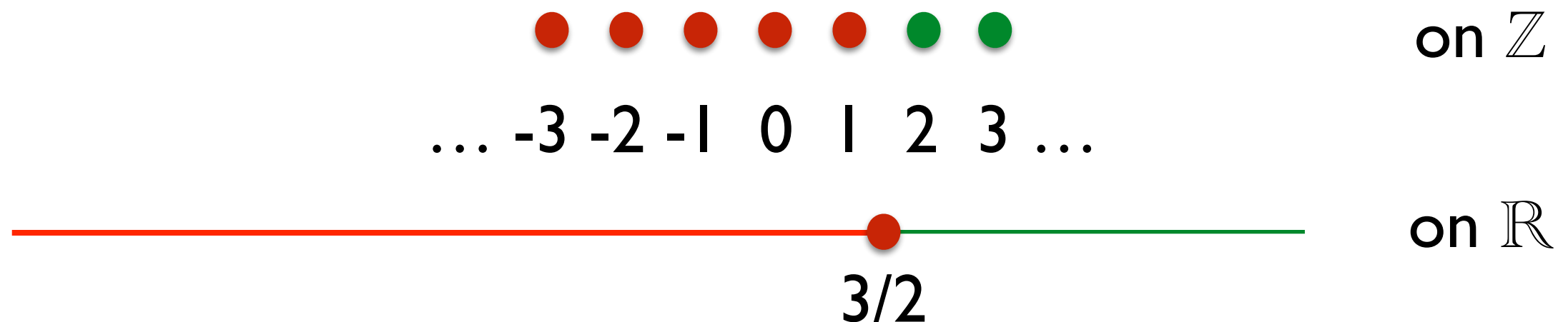this reasoning can not be expressed in propositional logic

Example

> Every player except the winner looses a match

Unary predicate (example)

Consider the statement 2m>3.

a unary relation

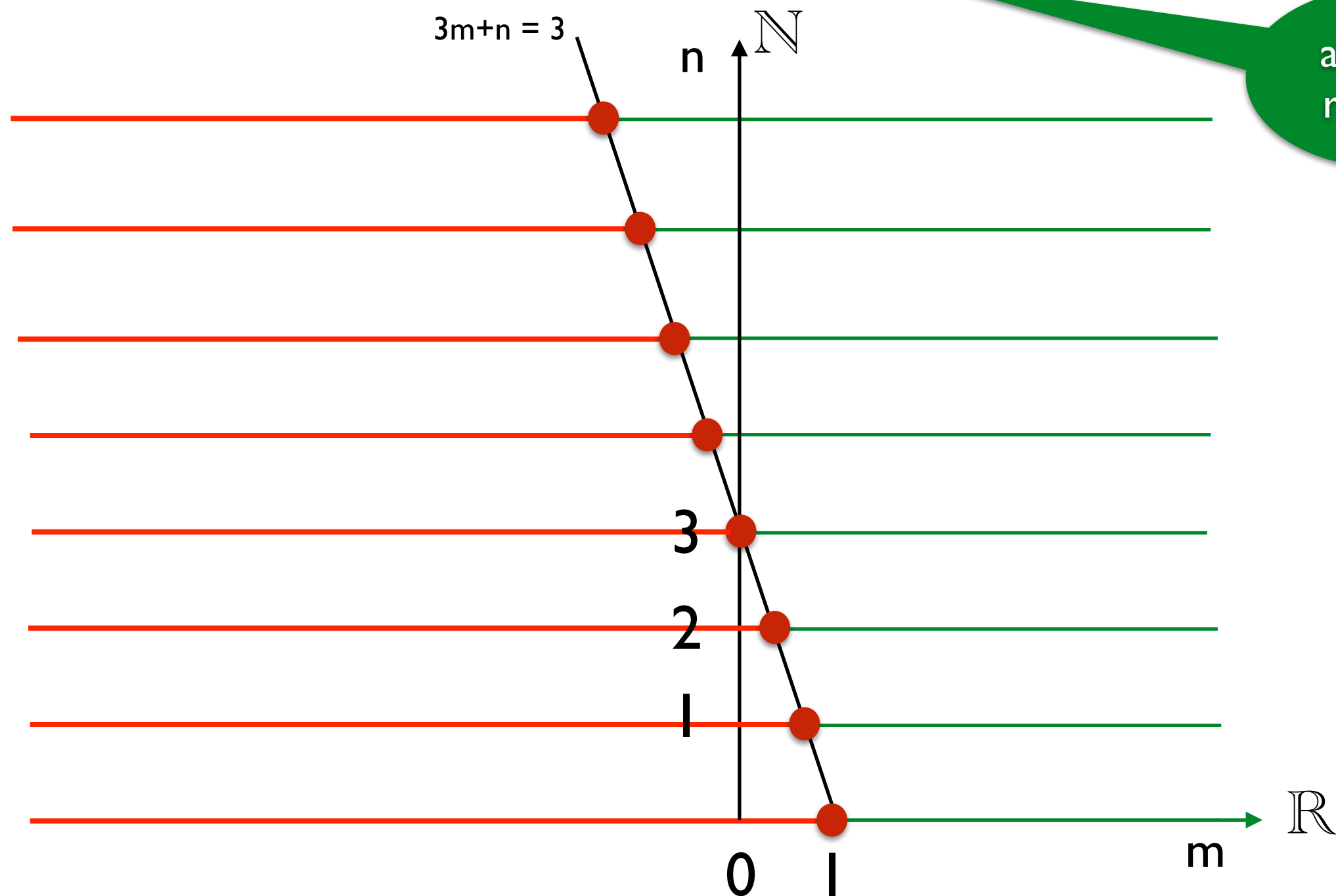Whether this statement is true or false depends on the value of m (and on the domain of values).

on $\mathbb{Z}$

... -3 -2 -1 0 1 2 3 ...

on $\mathbb{R}$

3/2

Note: $2m > 3 \overset{\text{val}}{=} m > 3/2$ on $\mathbb{Z}$ and $\mathbb{R}$

$2m > 3 \overset{\text{val}}{=} m \geq 2$ on $\mathbb{Z}$ but not on $\mathbb{R}$

Binary predicate (example)

The statement 3m+n > 3 is a binary predicate on $\mathbb{R}$ x $\mathbb{N}$.

Predicates

In general, an n-ary predicate is an n-ary relation.

If it is on a domain D, then it's a relation $P(x_1, .., x_n) \subseteq D^n$ or equivalently a function $P: D^n \to \{0,1\}$.

2m>3

true for certain values of the variables

We can turn a predicate, into a proposition in three ways:
1. By assigning values to the variables.
2. By universal quantification.
3. By existential quantification.

for m=2

$2 \cdot 2 > 3$

is a true proposition

Universal quantification

The unary predicate $2m > 3$ on $\mathbb{Z}$ can be turned into a proposition by universal quantification:

For all m in $\mathbb{Z}$, $2m > 3$

false, e.g. for m = 1

Notation: $\forall_m \, [m \in \mathbb{Z} : 2m > 3]$

universal quantifier

domain (predicate)

predicate

standard (!) notation:
$\forall x \, (P(x) \Rightarrow Q(x))$
$\forall x. \, P(x) \Rightarrow Q(x)$

In general: $\forall_x \, [P(x) : Q(x)]$ for "all x satisfying P satisfy Q"

Existential quantification

The unary predicate 2m > 3 on $\mathbb{Z}$ can also be turned into a proposition by existential quantification:

There exists m in $\mathbb{Z}$, 2m > 3

true, e.g. m = 2

existential quantifier

Notation: $\exists_m$ [m $\in$ $\mathbb{Z}$: 2m > 3]

domain (predicate)

predicate

standard (!) notation:
  $\exists x\ (P(x) \wedge Q(x))$
  $\exists x.\ P(x) \wedge Q(x)$

In general: $\exists_x$ [P(x) : Q(x)] for
"there exists x satisfying P that satisfies Q"

Notation

We write $\forall_x [P]$ for $\forall_x [T : P]$

also for $\exists$

We also write $\exists_{m,} \forall_n [(m,n) \in \mathbb{R} \times \mathbb{N} : 3m + n > 3]$

for $\exists_m [m \in \mathbb{R} : \forall_n [n \in \mathbb{N} : 3m + n > 3]]$

And even $\exists_{m, n} [(m,n) \in \mathbb{R} \times \mathbb{N} : 3m + n > 3]$

for $\exists_m [m \in \mathbb{R} : \exists_n [n \in \mathbb{N} : 3m + n > 3]]$

but only for the same quantifier!

Quantification - task

Let P be the set of all tennis players.
Let w ∈ P be the winner.

For p, q ∈ P, write p≠q for "p and q are different players" .

Let M be the set of all matches.
For p ∈ P and m ∈ M, write L(p,m) for
                                                    "player p loses match m".

Write the following sentence as a formula with predicates
and quantifiers:

Every player except the winner loses a match.