Transitive closure

Let R be a relation on a set X. The transitive closure (transitive Hülle) of R, notation R^+, is the relation

$$R^+ = \bigcup_{n \in \mathbb{N}, n \neq 0} R^n$$

$R^{n+1} = R^n \circ R$

The reflexive and transitive closure (reflexive und transitive Hülle) of R, notation R^*, is the relation

$$R^* = \bigcup_{n \in \mathbb{N}} R^n$$

$R^0 = \Delta_X$

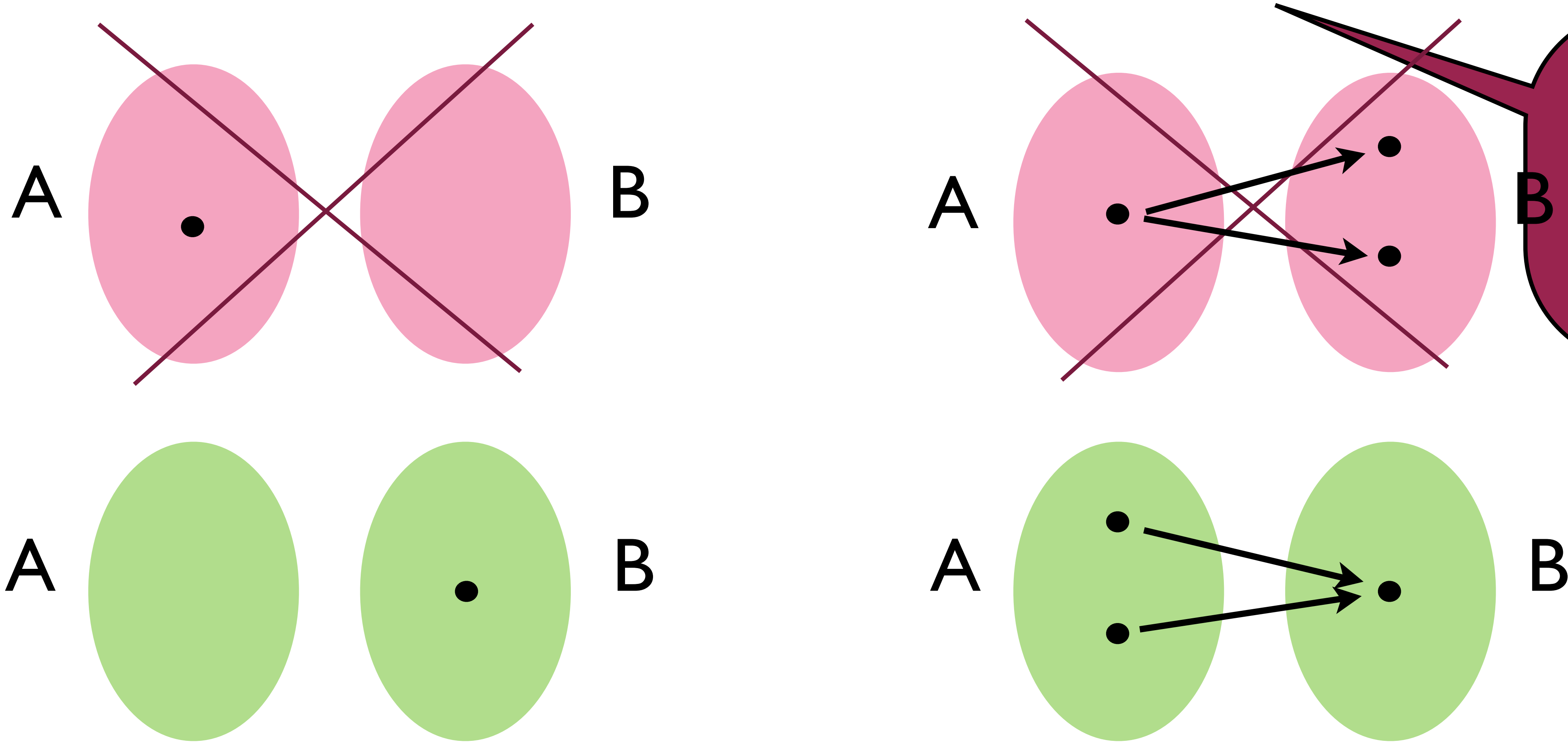
Proposition TC: Let R be a relation on X. The transitive closure of R is the smallest transitive relation that contains R. The reflexive and transitive closure of R is the smallest reflexive and transitive relation that contains R.

Functions, mappings

Def. If A and B are sets, then F is a function (mapping, Abbildung) from A to B, notation $F\colon A \longrightarrow B$ iff

for every $a \in A$, there exists a unique $b \in B$ such that $b = F(a)$.

the image of a,
$b = F(a)$
$a \mapsto b$



$\{(a, F(a)) \mid a \in A\}$ is the graph of the function F

Functions, mappings

When f: A ⟶ B then dom f = A and cod f = B

domain of F (Definitionsbereich)

codomain of F (Wertebereich)

Let f: A ⟶ B and A' ⊆ A.

The image (Bild) of A' is the set f(A') = {f(a) | a ∈ A'} ⊆ B.

f(A') = {b ∈ B | there is an a ∈ A' with b = f(a)}

if a ∈ A', then f(a) ∈ f(A')

So f extends to a function f: $\mathcal{P}$(A) ⟶ $\mathcal{P}$(B), the image-function.

Functions, mappings

Let f: $A \longrightarrow B$ and $B' \subseteq B$.

The inverse image (Urbild) of B' is the set

$$f^{-1}(B') = \{a \mid f(a) \in B'\} \subseteq A.$$

> $a \in f^{-1}(B')$ iff $f(a) \in B'$

Again the inverse image induces a function $f^{-1}: \mathcal{P}(B) \longrightarrow \mathcal{P}(A)$, the inverse-image-function.

Lemma F1: Let f: $A \longrightarrow B$, $A' \subseteq A$, and $B' \subseteq B$. Then

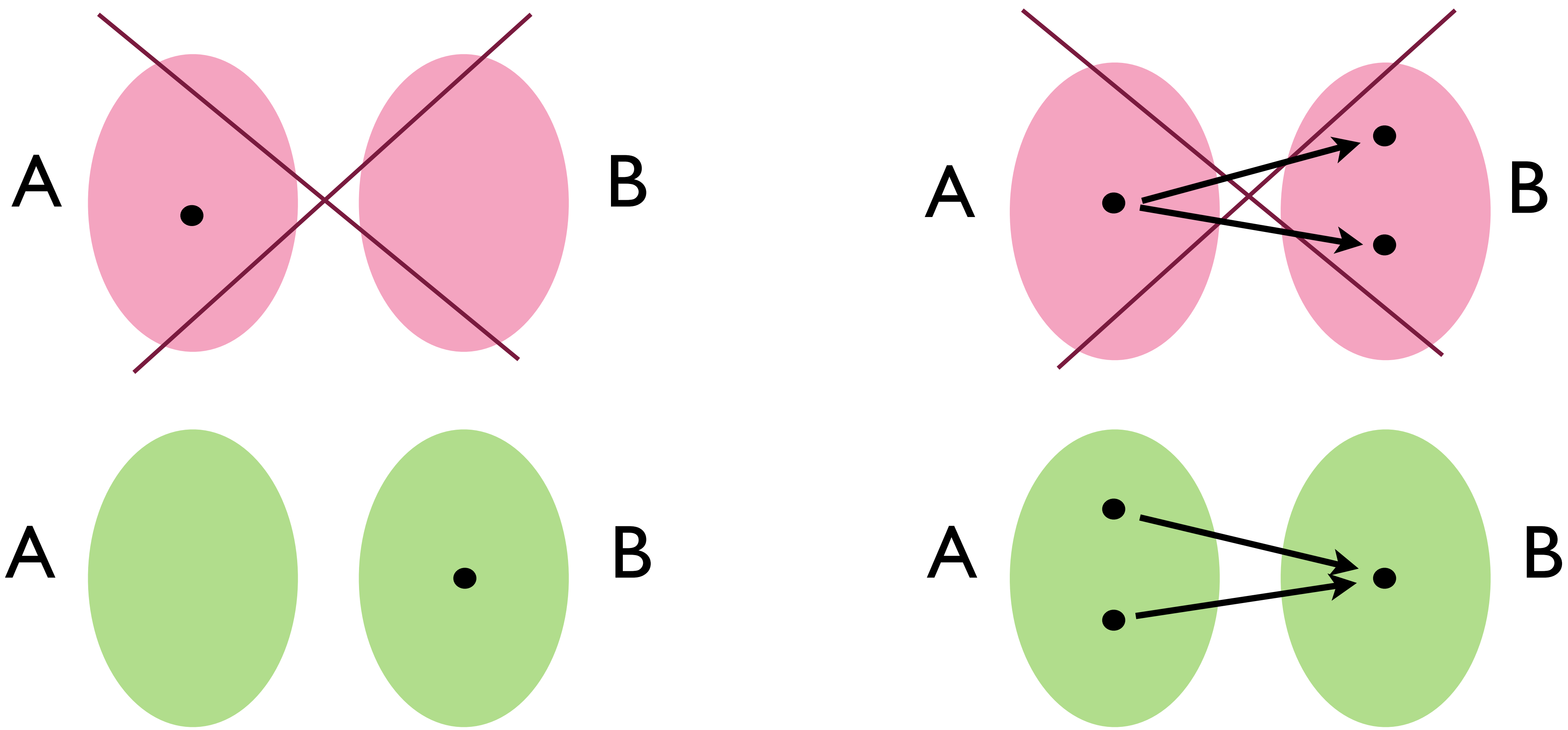
$$A' \subseteq f^{-1}(f(A')) \quad \text{and} \quad f(f^{-1}(B')) \subseteq B'$$

(in general no more than this holds)

Recall…

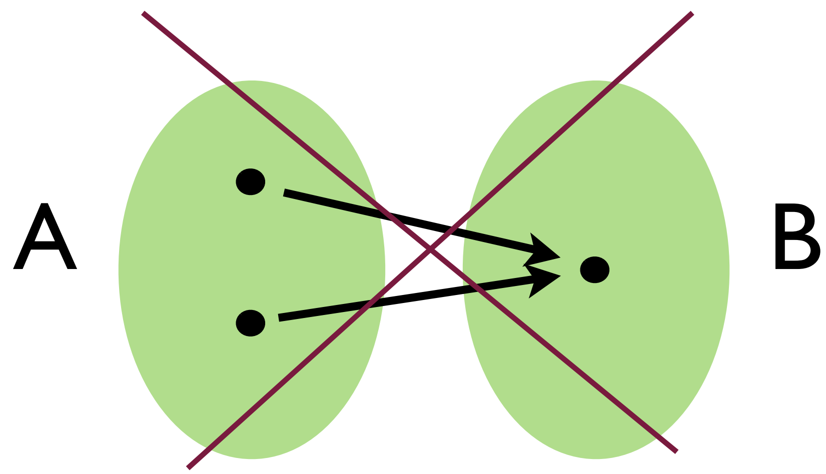
Def. If A and B are sets, then a relation $F \subseteq A \times B$ "is" a function (mapping, Abbildung) from A to B, notation $F: A \longrightarrow B$ iff

for every $a \in A$, there exists a unique $b \in B$ such that $b = F(a)$.

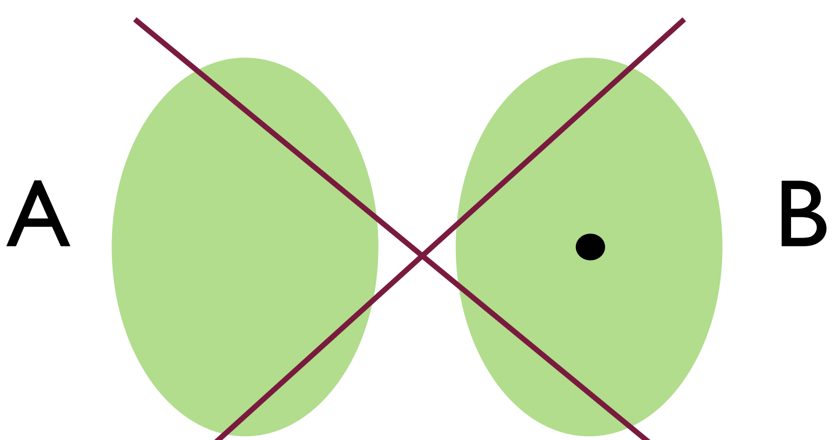


Special functions

The number of ingoing arrows for a function can be 0,1, or more.
Based on this, we distinguish some special functions.



at most one incoming arrow
injection



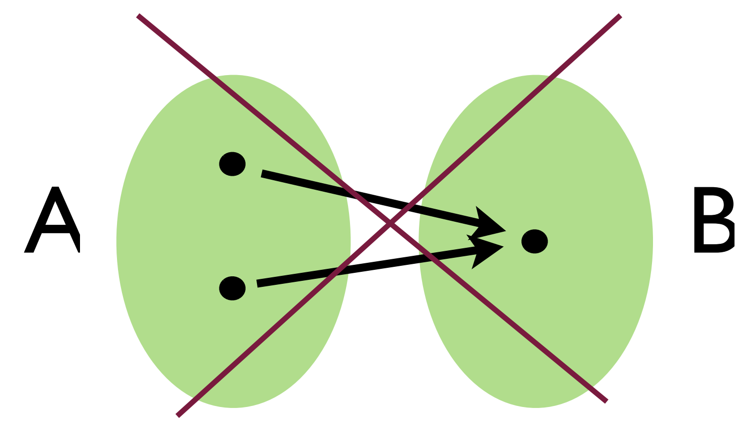
at least one incoming arrow
surjection

exactly one incoming arrow (injection + surjection) bijection

Special functions

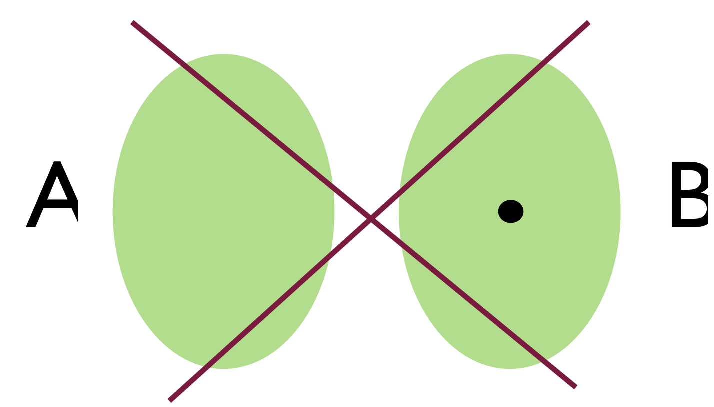
Def. A function f: A $\longrightarrow$ B is injective iff

for all a, b $\in$ A, if f(a) = f(b) then a = b.



Def. A function f: A $\longrightarrow$ B is surjective iff

for all b $\in$ B, there exists a $\in$ A such that f(a) = b.



Def. A function f: A $\longrightarrow$ B is bijective iff

for all b $\in$ B, there exists unique a $\in$ A with f(a) = b.

Simple characterisations

Lemma I1: A function $f: A \longrightarrow B$ is injective iff
for all $b \in B$, $|f^{-1}(\{b\})| \leq 1$.

at most one incoming arrow
injection

Lemma S1: A function $f: A \longrightarrow B$ is surjective iff
$|f^{-1}(\{b\})| \geq 1$ for all $b \in B$ iff
$f(A) = B$.

at least one incoming arrow
surjection

Lemma B1: A function $f: A \longrightarrow B$ is bijective iff
$|f^{-1}(\{b\})| = 1$ for all $b \in B$ iff
f is both injective and surjective.

exactly one incoming arrow
bijection

Some properties

Lemma I2: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f(x) \in f(A')$ iff $x \in A'$.

if holds always!

Prop. I3: Let $f: A \longrightarrow B$ be injective and let $A' \subseteq A$. Then

$f^{-1}(f(A')) = A'$.

Prop. S2: Let $f: A \longrightarrow B$ be surjective and let $B' \subseteq B$. Then

$f(f^{-1}(B')) = B'$.