Special relations

A relation $R \subseteq A \times A$ is:

reflexive	iff	for all $a \in A$, $(a,a) \in R$
symmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \in R$
transitive	iff	for all $a,b,c \in A$, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$
irreflexive	iff	for all $a \in A$, $(a,a) \notin R$
antisymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$
asymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \notin R$
total	iff	for all $a,b \in A$, $(a,b) \in R$ or $(b,a) \in R$

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(infix) notation aRb for $(a,b) \in R$

Special relations

A relation R on A, i.e., R $\subseteq$ A x A is:

equivalence	iff	R is reflexive, symmetric, and transitive
partial order	iff	R is reflexive, antisymmetric, and transitive
strict order	iff	R is irreflexive and transitive
preorder	iff	R is reflexive and transitive
total (linear) order	iff	R is a total partial order

Obvious properties

1. Every partial order is a preorder.

2. Every total order is a partial order.

3. Every total order is a preorder.

4. If $R \subseteq A \times A$ is a relation that contains cycles, i.e. there are $a, b \in A$ such that $a \neq b$, $(a,b) \in R$ and $(b,a) \in R$, then R is not a preorder, nor a partial order, nor a total order.

Operations on relations

Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be two relations. Their composition is the relation

$R \circ S = \{(a,c) \in A \times C \mid \text{there is } b \in B \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\}$

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Let $R \subseteq A \times B$ be a relation. The inverse relation of R is the relation

$$R^{-1} = \{(b,a) \in B \times A \mid (a, b) \in R\}$$

Characterizations

Lemma: Let R be a relation over the set A. Then

1. R is reflexive iff $\Delta_A \subseteq R$
2. R is symmetric iff $R \subseteq R^{-1}$
3. R is transitive iff $R^2 \subseteq R$

Important equivalence on $\mathbb{Z}$

Def. For a natural number n, the relation $\equiv_n$ is defined as

$i \equiv_n j$ iff $n \mid j - i$

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Lemma: The relation $\equiv_n$ is an equivalence for every n.

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Task: Describe the equivalence classes of $\equiv_n$
How many classes are there?