

Special relations

A relation $R \subseteq A \times A$ is:

reflexive	iff	for all $a \in A$, $(a,a) \in R$
symmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \in R$
transitive	iff	for all $a,b,c \in A$, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$
irreflexive	iff	for all $a \in A$, $(a,a) \notin R$
antisymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$
asymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \notin R$
total	iff	for all $a,b \in A$, $(a,b) \in R$ or $(b,a) \in R$

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(infix) notation aRb for $(a,b) \in R$

Special relations

A relation R on A , i.e., $R \subseteq A \times A$ is:

- equivalence iff R is reflexive, symmetric, and transitive
- partial order iff R is reflexive, antisymmetric, and transitive
- strict order iff R is irreflexive and transitive
- preorder iff R is reflexive and transitive
- total (linear)
order iff R is a total partial order

Obvious properties

1. Every partial order is a preorder.
2. Every total order is a partial order.
3. Every total order is a preorder.
4. If $R \subseteq A \times A$ is a relation that contains cycles,
i.e. there are $a, b \in A$ such that $a \neq b$, $(a,b) \in R$ and $(b,a) \in R$,
then R is not a preorder, nor a partial order, nor a total order.

Operations on relations

Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be two relations. Their composition is the relation

$$R \circ S = \{(a,c) \in A \times C \mid \text{there is } b \in B \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\}$$

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Let $R \subseteq A \times B$ be a relation. The inverse relation of R is the relation

$$R^{-1} = \{(b,a) \in B \times A \mid (a,b) \in R\}$$

Characterizations

Lemma: Let R be a relation over the set A . Then

1. R is reflexive iff $\Delta_A \subseteq R$
2. R is symmetric iff $R \subseteq R^{-1}$
3. R is transitive iff $R^2 \subseteq R$

Important equivalence on \mathbb{Z}

Def. For a natural number n , the relation \equiv_n is defined as

$$i \equiv_n j \quad \text{iff} \quad n \mid j - i$$

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[iff there exists $k \in \mathbb{Z}$ s.t. $j-i = k \cdot n$]

[iff $\exists k (k \in \mathbb{Z} \wedge j-i = k \cdot n)$]

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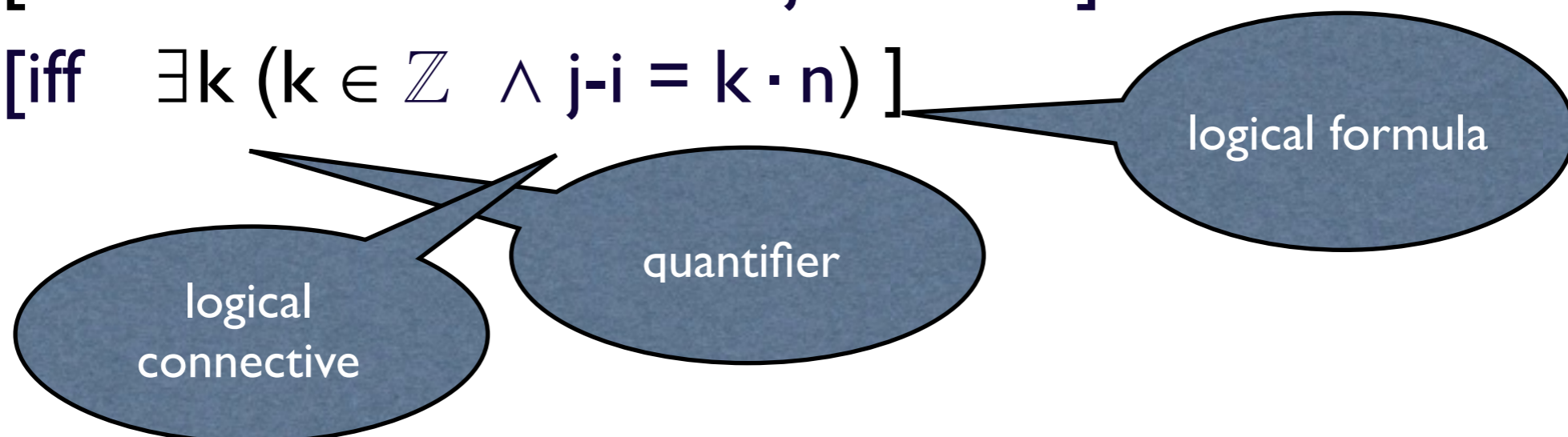
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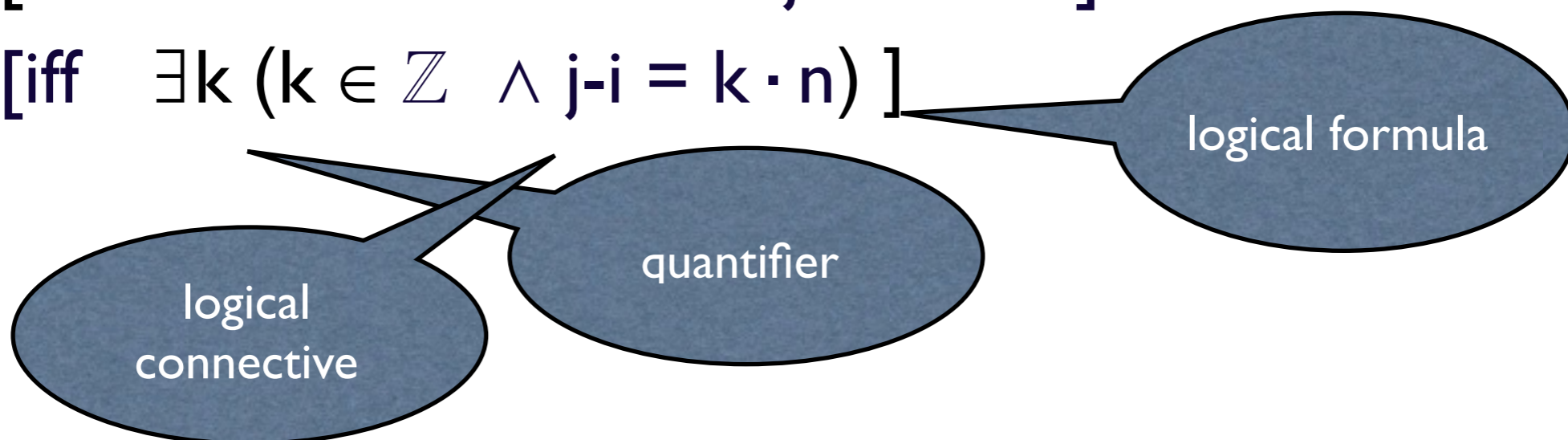
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Lemma: The relation \equiv_n is an equivalence for every n .

Equivalences classes

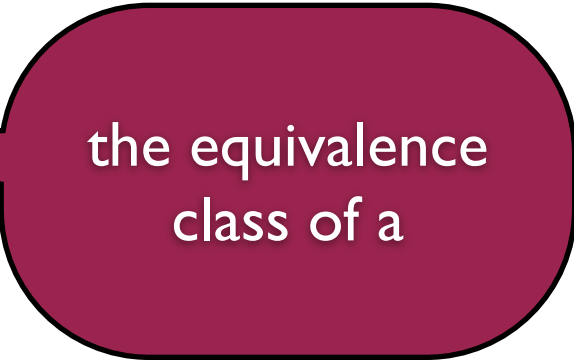
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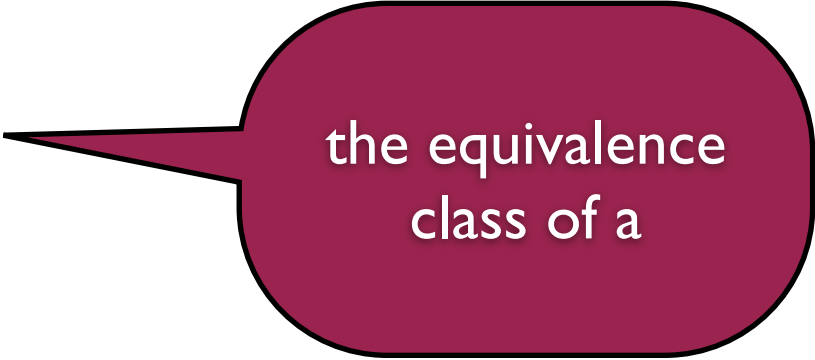


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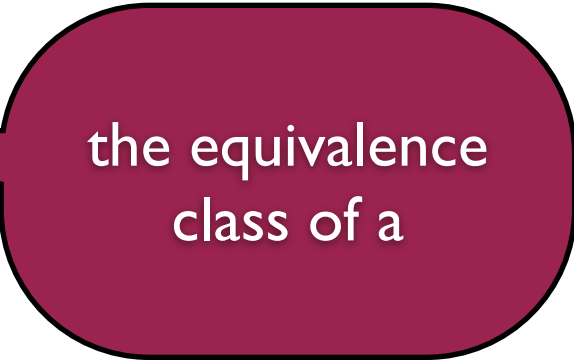
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Task: Describe the equivalence classes of \equiv_n
How many classes are there?