Regular languages and operations

$L(M_1) = \{w0 | w \in \{0,1\}^*\}$
is regular

Definition

Let Σ be an alphabet. A language L over Σ ($L \subseteq \Sigma^*$) is regular iff it is recognised by a DFA.

Regular operations

Let L, L_1, L_2 be languages over Σ. Then $L_1 \cup L_2$, $L_1 \cdot L_2$, and L^* are languages, where

$$L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\}$$

$$L^* = \{w \mid \exists n \in \mathbb{N}.\ \exists\, w_1, w_2, .., w_n \in L.\ w = w_1 w_2 .. w_n\}$$

$\varepsilon \in L^*$ always

Closure under regular operations

also under intersection

Theorem C1

The class of regular languages is closed under union

We can already prove these!

Theorem C2

The class of regular languages is closed under complement

Theorem C3

The class of regular languages is closed under concatenation

But not yet these two…

Theorem C4

The class of regular languages is closed under Kleene star

Nondeterministic Automata (NFA)

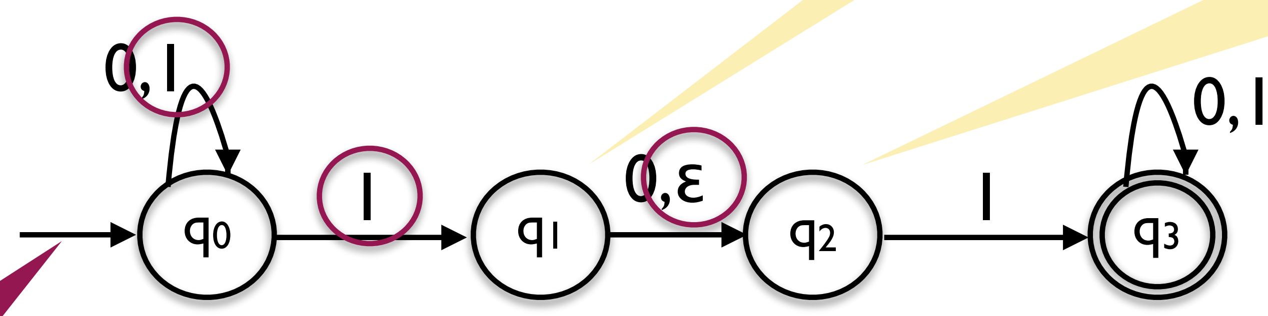
no 1 transition

Informal example

no 0 transition

$\Sigma = \{0,1\}$

M_2:



sources of nondeterminism

Accepts a word iff there exists an accepting run

NFA

Definition

A nondeterministic automaton M is a tuple $M = (Q, \Sigma, \delta, q_0, F)$ where

- Q is a finite set of states
- Σ is a finite alphabet
- $\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ is the transition function
- q_0 is the initial state, $q_0 \in Q$
- F is a set of final states, $F \subseteq Q$

$\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

In the example M

$M_2 = (Q, \Sigma, \delta, q_0, F)$ for

$Q = \{q_0, q_1, q_2, q_3\}$

$\Sigma = \{0, 1\}$ $F = \{q_3\}$

$\delta(q_0, 0) = \{q_0\}$
$\delta(q_0, 1) = \{q_0, q_1\}$
$\delta(q_0, \varepsilon) = \{q_0\}$
.....

ε-closure of q, all states reachable by ε-transitions from q

NFA

$E(q) = \{q' \mid q' = q \vee \exists n \in \mathbb{N}^+ . \exists q_0, .., q_n \in Q . q_0 = q, q_n = q', q_{i+1} \in \delta(q_i, \varepsilon), \text{ for } i = 0, .., n-1\}$

The extended transition function

Given an NFA M = $(Q, \Sigma, \delta, q_0, F)$ we can extend $\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ to

$\delta^*: Q \times \Sigma^* \longrightarrow \mathcal{P}(Q)$

$E(X) = \bigcup_{x \in X} E(x)$

In M_2, $\delta^*(q_0, 0110) = \{q_0, q_2, q_3\}$

inductively, by:

$\delta^*(q, \varepsilon) = E(q)$ and $\delta^*(q, wa) = E(\bigcup_{q' \in \delta^*(q,w)} \delta(q', a))$

Definition

$L(M_2) = \{u101w \mid u,w \in \{0,1\}^*\} \cup \{u11w \mid u,w \in \{0,1\}^*\}$

The language recognised / accepted by a ... automaton M = $(Q, \Sigma, \delta, q_0, F)$ is

$$L(M) = \{w \in \Sigma^* \mid \delta^*(q_0, w) \cap F \neq \emptyset\}$$

Equivalence of automata

Definition

Two automata M_1 and M_2 are equivalent if $L(M_1) = L(M_2)$

Theorem NFA ~ DFA

Every NFA has an equivalent DFA

Proof via the "powerset construction" / determinization

Corollary

A language is regular iff it is recognised by a NFA