Infinite, countable and uncountable sets

We write $\aleph_0$ for the cardinality of natural numbers.
Hence $\aleph_0 = |\mathbb{N}|$.

$|A| = [A]_\sim$

cardinal numbers are $\sim$ equivalence classes

Def. A set A is countable iff $|A| = \aleph_0$.

Prop. $\mathbb{N}$ is countable.
$\mathbb{Z}$ is countable.
$\mathbb{Q}$ is countable.

Def. A set is infinite iff $|A| \geq \aleph_0$.

Hence, every countable set is infinite

Def. A set is uncountable iff $|A| > \aleph_0$.

Prop. $\mathbb{R}$ is uncountable.

We write c for $|\mathbb{R}|$

Cardinals are unbounded

Theorem (Cantor)

For every set A we have $|A| < |\mathcal{P}(A)|$.

Hence, for every cardinal there is a larger one.

$|A| = [A]_\sim$

cardinal numbers are $\sim$ equivalence classes

Finite Automata

Alphabets and Languages

Recall

$\Sigma^0 = \{\varepsilon\}$ contains only the empty word

Σ - alphabet (finite set)

$\Sigma^n = \{a_1a_2..a_n \mid a_i \in \Sigma\}$ is the set of words of length n

$\Sigma^* = \{w \mid \exists n \in \mathbb{N}.\ \exists\ a_1, a_2, .., a_n \in \Sigma.\ w = a_1a_2..a_n\}$ is the set of all words over Σ

A language L over Σ is a subset $L \subseteq \Sigma^*$

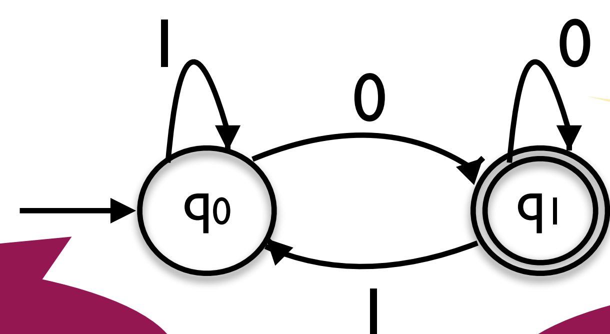
Deterministic Automata (DFA)

alphabet

Informal example

$\Sigma = \{0,1\}$

M_1:



q_0, q_1 are states

transitions, labelled by alphabet symbols

q_0 is initial

q_1 is final

Accepts the language $L(M_1) = \{w \in \Sigma^* \mid w \text{ ends with a } 0\} = \Sigma^* 0$

regular language

regular expression

DFA

Definition

A deterministic automaton M is a tuple $M = (Q, \Sigma, \delta, q_0, F)$ where

Q is a finite set of states
Σ is a finite alphabet
$\delta: Q \times \Sigma \longrightarrow Q$ is the transition function
q_0 is the initial state, $q_0 \in Q$
F is a set of final states, $F \subseteq Q$

In the example M

$M_1 = (Q, \Sigma, \delta, q_0, F)$ for

$Q = \{q_0, q_1\}$ $F = \{q_1\}$

$\Sigma = \{0, 1\}$

$\delta(q_0, 0) = q_1$,$\delta(q_0, 1) = q_0$
$\delta(q_1, 0) = q_1$, $\delta(q_1, 1) = q_0$

DFA

The extended transition function

Given $M = (Q, \Sigma, \delta, q_0, F)$ we can extend $\delta: Q \times \Sigma \longrightarrow Q$ to

$\delta^*: Q \times \Sigma^* \longrightarrow Q$

inductively, by:

$\delta^*(q, \varepsilon) = q$ and $\delta^*(q,wa) = \delta(\delta^*(q,w), a)$

In M_1, $\delta^*(q_0, 110010) = q_1$

Definition

The language recognised / accepted by a deterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ is

$$L(M) = \{w \in \Sigma^* \mid \delta^*(q_0,w) \in F\}$$

$L(M_1) = \{w0 \mid w \in \{0,1\}^*\}$