Proofs with ∃-introduction and ∃-elimination are unnecessarily long and cumbersome…

There are alternatives!

Proving an existential quantification

To prove

$\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$

Proof

It suffices to find a witness, i.e., an $x \in \mathbb{Z}$ satisfying

$$x^3 - 2x - 8 \geq 0.$$

$x = 3$ is a witness, since $3 \in \mathbb{Z}$ and $3^3 - 2 \cdot 3 - 8 = 13 \geq 0$

Conclusion: $\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$.

also x = 5 is a witness…

Alternative ∃ introduction

How do we prove an existential quantification?

by finding
a witness

∃

...

(k) P(a)

...

(l) Q(a)

...

{∃*-intro on (k) and (l)}

(m) ∃x [P(x) : Q(x)]

strategy: wait until a witness object appears

does not always work

(k < m, l < m)

Using an existential quantification

We know $\exists x[x \in \mathbb{R} : a - x < 0 < b - x]$

We can declare an $x \in \mathbb{Z}$ (a witness) such that

$$a - x < 0 < b - x$$

and use it further in the proof. For example:

From $a - x < 0$, we get $a < x$.

From $b - x > 0$, we get $x < b$.

Hence, $a < b$.

Alternative ∃ elimination

How do we use an existential quantification in a proof?

we pick a witness

∃

|| ||

(k) ∃x [P(x) : Q(x)]

|| ||

{∃*-elim on (k)}

(m) Pick x with P(x) and Q(x)

x must be new!

time for an example!

(k < m)