

Negation introduction

How do we prove a negation?

$$\neg P \stackrel{\text{val}}{=} P \Rightarrow F$$

\Rightarrow -intro

\neg -introduction

...

{Assume}

(k) P

...

(l-1) F

{ \neg -intro on (k) and (l-1)}

(l) $\neg P$

Negation elimination

How do we use a negation in a proof?

$$P \wedge \neg P \stackrel{\text{val}}{=} F$$

\neg -elimination

(k)	P
(l)	$\neg P$
	{ \neg -elim on (k) and (l)}
(m)	F

$(k < m, l < m)$

time for an example!

F introduction

How do we prove F?

$$P \wedge \neg P \stackrel{\text{val}}{=} F$$

F-introduction

...

(k) P

...

(l) $\neg P$

...

{F-intro on (k) and (l)}

(m) F

(k < m, l < m)

the same as \neg -elim
only intended bottom-up

F elimination

How do we use F in a proof?

it's very useful!

F-elimination

|| |
(k) F
|| |
{F-elim on (k)}
(m) P

(k < m)

$F \stackrel{\text{val}}{\Vdash} P$

Double negation introduction

How do we prove $\neg\neg P$?

$\neg\neg$ -introduction

...

(k) P

...

{ $\neg\neg$ -intro on (k)}

(m) $\neg\neg P$

(k < m)

$P \stackrel{\text{val}}{=} \neg\neg P$

Double negation elimination

How do we use $\neg\neg$ in a proof?

$\neg\neg$ -elimination

|| |
(k) $\neg\neg P$
|| |
{ $\neg\neg$ -elim on (k)}
(m) P

(k < m)

$\neg\neg P \stackrel{\text{val}}{=} P$

Proof by contradiction

Theorem

If x^2 is even, then x is even ($x \in \mathbb{Z}$).

Proof

Let $x \in \mathbb{Z}$

Assume x^2 is even.

Assume that x is odd.

Then $x = 2y+1$ for some $y \in \mathbb{Z}$.

Then $x^2 = (2y+1)^2 = 4y^2 + 4y + 1 = 2(2y^2 + 2y) + 1$ and $2y^2 + 2y \in \mathbb{Z}$.

So, x^2 is odd

a contradiction.

So, x is even

(sub)goal

generating hypothesis

pure hypothesis

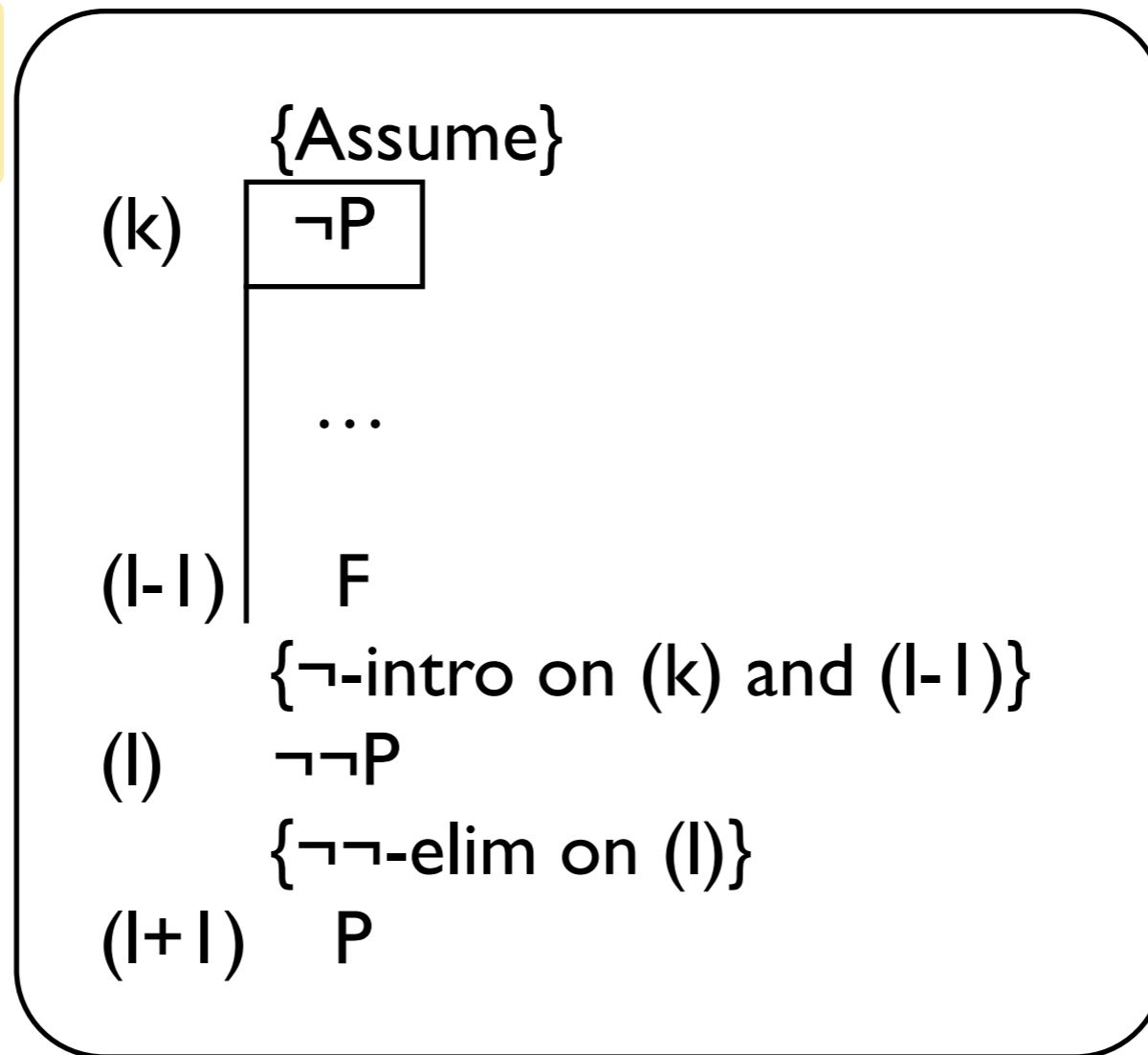
conclusion

Thanks to Bas Luttik

Proof by contradiction

How do we prove P by a contradiction?

proof by contradiction



(k < m)

$$\neg P \Rightarrow F \stackrel{\text{val}}{=} \neg\neg P \stackrel{\text{val}}{=} P$$

\neg -intro

$\neg\neg$ -elim

time for an example!

Disjunction introduction

How do we prove a disjunction?

$$\neg P \Rightarrow Q \stackrel{\text{val}}{\vDash} P \vee Q$$

$$\neg Q \Rightarrow P \stackrel{\text{val}}{\vDash} P \vee Q$$

\Rightarrow -intro

v-introduction

...

{Assume}

(k) $\neg P$

...

(l-1) Q

{v-intro on (k) and (l-1)}

(l) $P \vee Q$

Disjunction introduction

How do we prove a disjunction?

$$\neg P \Rightarrow Q \stackrel{\text{val}}{\vDash} P \vee Q$$

$$\neg Q \Rightarrow P \stackrel{\text{val}}{\vDash} P \vee Q$$

\Rightarrow -intro

\vee -introduction

...

{Assume}

(k) $\neg Q$

...

(l-1) P

{ \vee -intro on (k) and (l-1)}

(l) $P \vee Q$

Disjunction elimination

How do we use a disjunction in a proof?

$$P \vee Q \stackrel{\text{val}}{\models} \neg P \Rightarrow Q$$

$$P \vee Q \stackrel{\text{val}}{\models} \neg Q \Rightarrow P$$

v-elimination

|||

(k) $P \vee Q$

|||

{v-elim on (k)}

(m) $\neg P \Rightarrow Q$

(k < m)

Disjunction elimination

How do we use a disjunction in a proof?

$$P \vee Q \stackrel{\text{val}}{\models} \neg P \Rightarrow Q$$

$$P \vee Q \stackrel{\text{val}}{\models} \neg Q \Rightarrow P$$

v-elimination

|| |

(k) $P \vee Q$

|| |

{v-elim on (k)}

(m) $\neg Q \Rightarrow P$

(k < m)

Proof by case distinction

How do we prove R by a case distinction?

proof by
case distinction

|| |
(k) $P \vee Q$

|| |
(l) $P \Rightarrow R$

|| |
(m) $Q \Rightarrow R$

|| |
(n) $\{ \text{case-dist on } (k), (l), (m) \}$
R

$(k < n, l < n, m < n)$

$$(P \vee Q) \wedge (P \Rightarrow R) \wedge (Q \Rightarrow R) \stackrel{\text{val}}{\vDash} R$$

Bi-implication introduction

How do we prove a bi-implication?

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P) \stackrel{\text{val}}{=} P \Leftrightarrow Q$$

\Leftrightarrow -introduction

...

(k) $P \Rightarrow Q$

...

(l) $Q \Rightarrow P$

...

{ \Leftrightarrow -intro on (k) and (l)}

(m) $P \Leftrightarrow Q$

$(k < m, l < m)$

\wedge -intro

Bi-implication elimination

How do we use a bi-implication in a proof?

\Leftrightarrow -elimination

$$P \Leftrightarrow Q \stackrel{\text{val}}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

|||
(k) $P \Leftrightarrow Q$

|||
{ \Leftrightarrow -elim on (k)}
(m) $P \Rightarrow Q$

(k < m)

|||
(k) $P \Leftrightarrow Q$

|||
{ \Leftrightarrow -elim on (k)}
(m) $Q \Rightarrow P$

(k < m)

\wedge -elim

Derivations / Reasoning with quantifiers

Proving a universal quantification

To prove

$$\forall x[x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$$

Proof

Let $x \in \mathbb{Z}$ be arbitrary and assume that $x \geq 2$.

Then, for this particular x , it holds that

$$x^2 - 2x = x(x-2) \geq 0 \quad (\text{Why?})$$

Conclusion: $\forall x[x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$.

\forall introduction

How do we prove a universal quantification?

similar to
 \Rightarrow -intro
with
generating
hypothesis

\forall -introduction

...

{Assume}

(k) **var x; P(x)**

...

(l-1) Q(x)
{ \forall -intro on (k) and (l-1)}

(l) $\forall x[P(x) : Q(x)]$

flag shows the validity of a hypothesis

Using a universal quantification

We know

$$\forall x [x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$$

Whenever we encounter an $a \in \mathbb{Z}$ such that $a \geq 2$,
we can conclude that $a^2 - 2a \geq 0$.

For example, $(52387^2 - 2 \cdot 52387) \geq 0$
since $52387 \in \mathbb{Z}$ and $52387 \geq 2$.

\forall elimination

How do we use a universal quantification in a proof?

similar to implication but we need a witness

\forall -elimination

|| |
(k) $\forall x[P(x) : Q(x)]$

|| |
(l) $P(a)$

|| |
{ \forall -elim on (k) and (l)}
(m) $Q(a)$

a is an object (variable, number,..) which is "known" in line (l)

the same "a" from line (l)

time for an example!

$(k < m, l < m)$

\exists introduction

How do we prove an existential quantification?

$$\neg \forall x [P(x) : \neg Q(x)] \stackrel{\text{val}}{=} \exists x [P(x) : Q(x)]$$

\exists -introduction

...

{Assume}

(k) $\forall x [P(x) : \neg Q(x)]$

...

(l-1) F

{ \exists -intro on (k) and (l-1)}

(l) $\exists x [P(x) : Q(x)]$

and \neg -intro

\exists elimination

How do we use an existential quantification in a proof?

\exists -elimination

|| |
(k) $\exists x [P(x) : Q(x)]$
|| |
(l) $\forall x [P(x) : \neg Q(x)]$
|| |
{ \exists -elim on (k) and (l)}
(m) F

(k < m, l < m)

$\exists x [P(x) : Q(x)] \stackrel{\text{val}}{\models} \neg \forall x [P(x) : \neg Q(x)]$

and \neg -
elimination

time for an
example!

Proofs with \exists -introduction and \exists -elimination are unnecessarily long and cumbersome...



There are alternatives!

Alternative \exists elimination

How do we use an existential quantification in a proof?

we pick a witness

\exists

|| |

(k) $\exists x [P(x) : Q(x)]$

|| |

{ \exists -elim on (k)}

(m) Pick x with $P(x)$ and $Q(x)$

x must be new!

time for an example!

(k < m)