Predicate logic

Limitations of propositional logic

Propositional logic only allows us to reason about completed statements about things, not about the things themselves.

Example

Some chicken cannot fly
All chicken are birds

Some birds cannot fly

this reasoning can not be expressed in propositional logic

Example

Every player except the winner looses a match

Unary predicate (example)

Consider the statement $2m>3$.

Whether this statement is true or false depends on the value of m (and on the domain of values).

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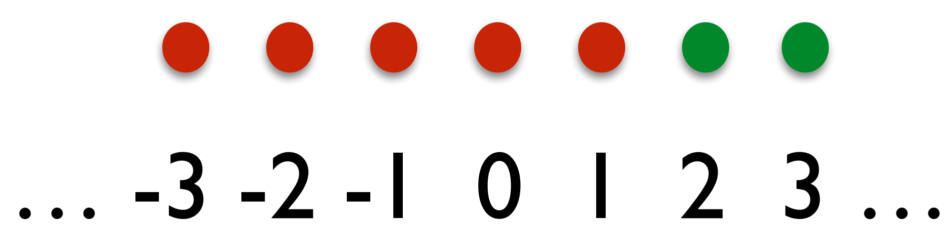

on $\mathbb{Z}$

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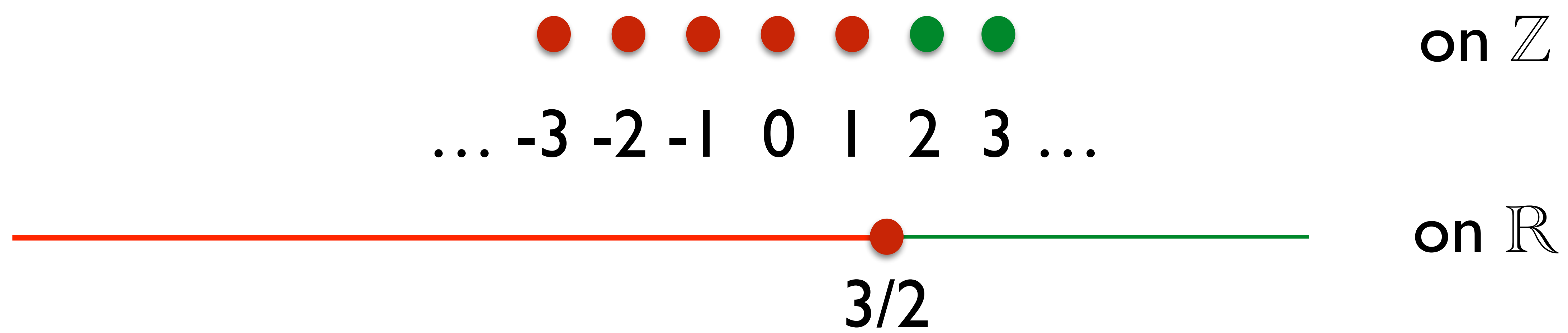

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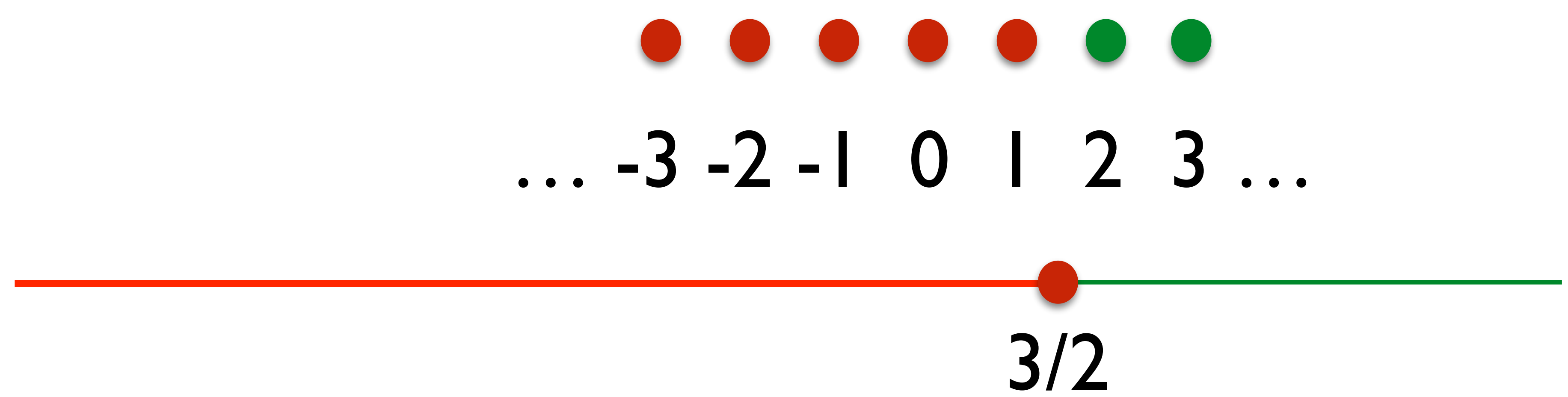

on $\mathbb{Z}$

on $\mathbb{R}$

Note: $2m > 3 \overset{\text{val}}{=} m > 3/2$ on $\mathbb{Z}$ and $\mathbb{R}$

$2m > 3 \overset{\text{val}}{=} m \geq 2$ on $\mathbb{Z}$ but not on $\mathbb{R}$

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a unary relation

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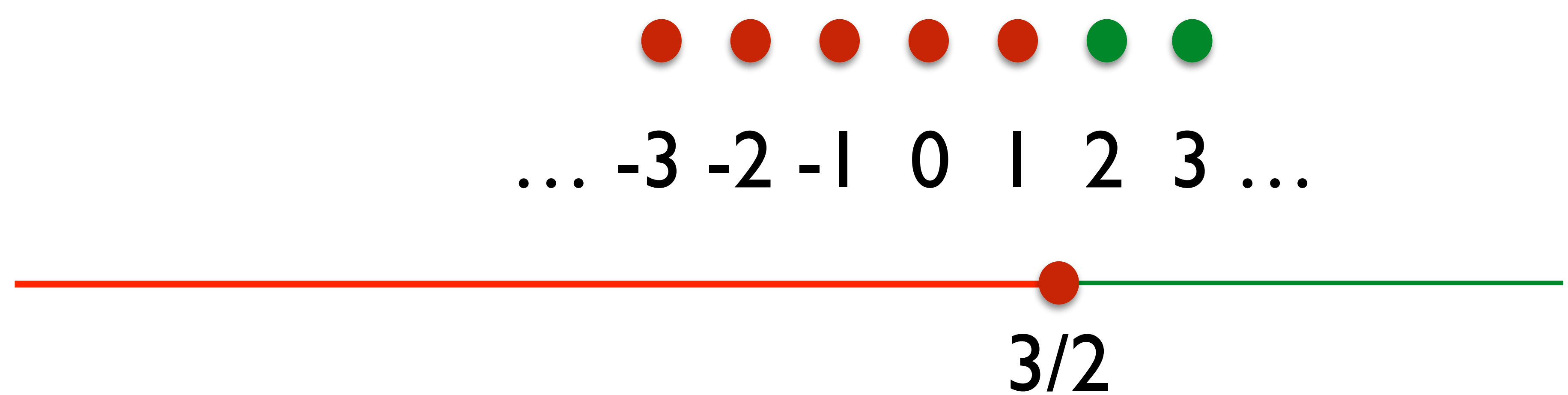

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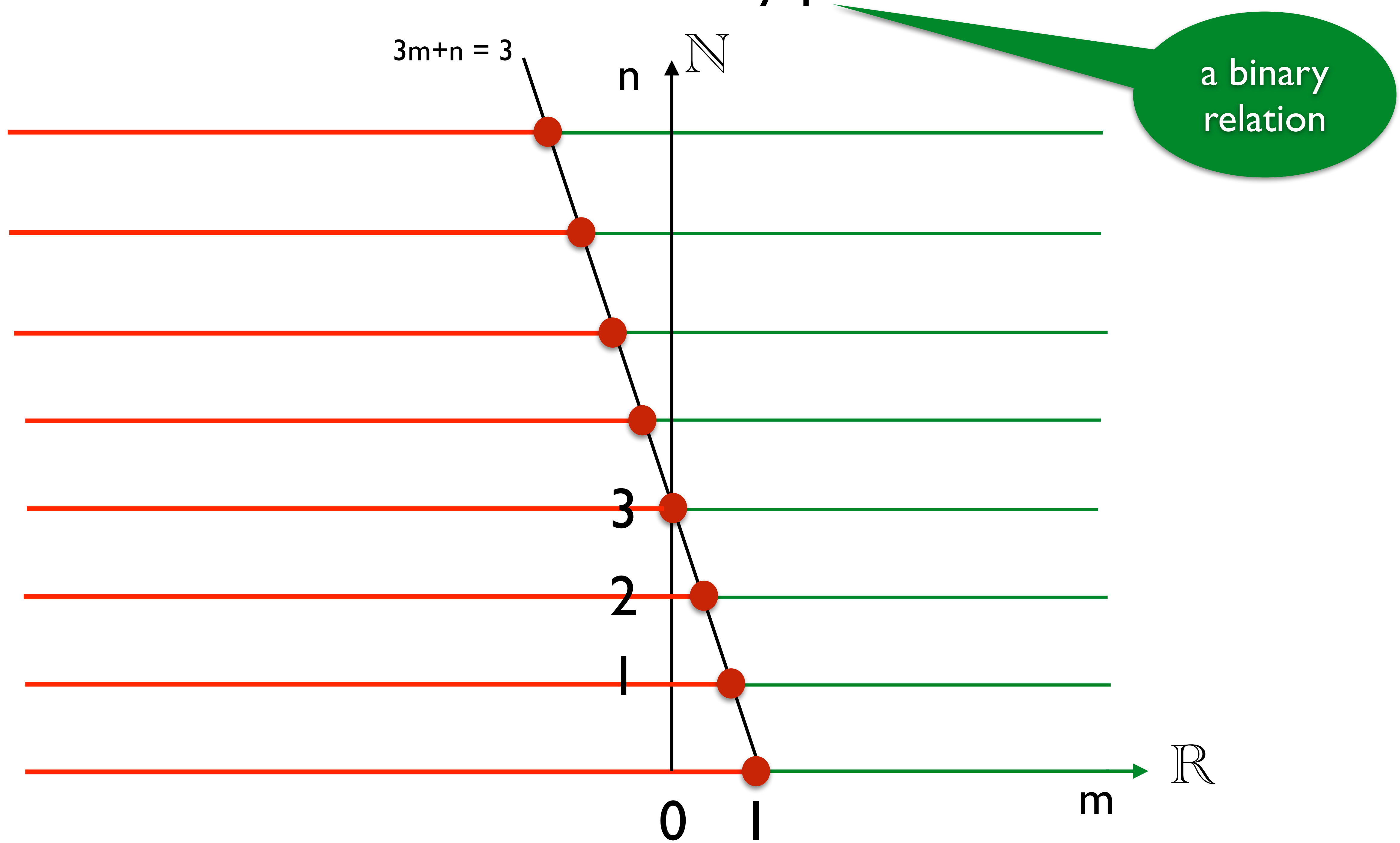
Binary predicate (example)

The statement $3m+n > 3$ is a binary predicate on $\mathbb{R} \times \mathbb{N}$.



Predicates

In general, an n-ary predicate is an n-ary relation.

If it is on a domain D, then it's a relation $P(x_1, .., x_n) \subseteq D^n$ or equivalently a function $P: D^n \rightarrow \{0,1\}$.

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We can turn a predicate, into a proposition in three ways:

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for $m=2$

$2 \cdot 2 > 3$

is a true proposition

Universal quantification

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The binary predicate $3m+n > 3$ on $\mathbb{R}$ x $\mathbb{N}$ can also be turned into a proposition by quantification:

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unary predicate

binary predicate

proposition, nullary predicate

other notation:

$\exists m.\ (m \in \mathbb{R} \wedge \forall n.\ (n \in \mathbb{N} \Rightarrow 3m+n>3))$

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We also write $\exists_{m,} \forall_n$ [(m,n)$\in \mathbb{R}$ x $\mathbb{N}$: 3m + n >3]

for $\exists_m$ [m$\in \mathbb{R}$: $\forall_n$ [n$\in \mathbb{N}$: 3m + n >3]]

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And even $\exists_{m,\,n}\ [(m,n) \in \mathbb{R} \times \mathbb{N} : 3m + n > 3]$

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but only for the same quantifier!

Quantification - task

Let P be the set of all tennis players.
Let $w \in P$ be the winner.

For $p, q \in P$, write $p \neq q$ for "p and q are different players".

Let M be the set of all matches.
For $p \in P$ and $m \in M$, write $L(p,m)$ for
"player p loses match m".

Write the following sentence as a formula with predicates and quantifiers:

Every player except the winner loses a match.

Quantification - task

Thanks to Bas Luttik

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Equivalences with quantifiers

Renaming bound variables

Bound variables

$$\forall_x[P{:}Q] \stackrel{val}{=} \forall_y[P[y/x]{:}Q[y/x]]$$

$$\exists_x[P{:}Q] \stackrel{val}{=} \exists_y[P[y/x]{:}Q[y/x]]$$

if y does not occur in
P or Q (not even in ∀y, ∃y)

Domain splitting

Examples:

$$\forall_x[x \leqslant 1 \vee x \geqslant 5\colon x^2 - 6x + 5 \geqslant 0]$$
$$\stackrel{val}{=} \forall_x[x \leqslant 1\colon x^2 - 6x + 5 \geqslant 0] \wedge \forall_x[x \geqslant 5\colon x^2 - 6x + 5 \geqslant 0]$$

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$$\begin{aligned} & \exists_k[0 \leqslant k \leqslant n : k^2 \leqslant 10] \\ \stackrel{val}{=} \; & \exists_k[0 \leqslant k \leqslant n-1 \vee k = n : k^2 \leqslant 10] \\ \stackrel{val}{=} \; & \exists_k[0 \leqslant k \leqslant n-1 : k^2 \leqslant 10] \vee \exists_k[k = n : k^2 \leqslant 10] \end{aligned}$$

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Domain splitting

$$\forall_x[P \vee Q : R] \stackrel{val}{=} \forall_x[P : R] \wedge \forall_x[Q : R]$$
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Equivalences with quantifiers

One-element domain

$$\forall_x [x = n : Q] \overset{val}{=} Q[n/x]$$

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Empty domain

$$\forall_x[F : Q] \overset{val}{=} T$$

$$\exists_x[F : Q] \overset{val}{=} F$$

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"All Marsians are green"

Domain weakening

Intuition: The following are equivalent

$$\forall_x[x \in D : A(x)] \quad \text{and} \quad \forall_x[x \in D \Rightarrow A(x)]$$
$$\exists_x[x \in D : A(x)] \quad \text{and} \quad \exists_x[x \in D \wedge A(x)]$$

The same can be done to parts of the domain

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Domain weakening

$$\forall_x [P \wedge Q : R] \stackrel{val}{=} \forall_x [P : Q \Rightarrow R]$$
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$$P \wedge Q \stackrel{val}{\models} P$$

De Morgan with quantifiers

De Morgan

$$\neg\forall_x[P:Q] \stackrel{val}{=} \exists_x[P:\neg Q]$$
$$\neg\exists_x[P:Q] \stackrel{val}{=} \forall_x[P:\neg Q]$$

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Hence: $\neg \forall = \exists \neg$ and $\neg \exists = \forall \neg$

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Hence: $\neg\forall = \exists\neg$ and $\neg\exists = \forall\neg$

It holds further that:

$$\neg\forall_x\neg = \exists_x\neg\neg = \exists_x$$
$$\neg\exists_x\neg = \forall_x\neg\neg = \forall_x$$

Substitution

meta rule

Simple

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P] \overset{val}{=} \psi[\xi/P]}$$

EVERY occurrence of P is substituted!

Sequential

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{=} \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\phi \overset{val}{=} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{=} \psi[\xi/P, \eta/Q]}$$

holds also for quantified formulas!

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The rule of Leibnitz

meta rule

Leibnitz

$$\frac{\phi \overset{val}{=} \psi}{C[\phi] \overset{val}{=} C[\psi]}$$

formula that has ϕ as a sub formula

single occurrence is replaced!

holds also for
quantified formulas!

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meta rule

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Other equivalences with quantifiers

Exchange trick

$$\forall_x[P{:}Q] \overset{val}{=} \forall_x[\neg Q{:}\neg P]$$

$$\exists_x[P{:}Q] \overset{val}{=} \exists_x[Q{:}P]$$

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No wonder as

$$\forall_x[P{:}Q] \stackrel{val}{=} \forall_x[P \Rightarrow Q]$$

$$\exists_x[P{:}Q] \stackrel{val}{=} \exists_x[P \wedge Q]$$

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Term splitting

$$\forall_x[P{:}Q \wedge R] \stackrel{val}{=} \forall_x[P{:}Q] \wedge \forall_x[P{:}R]$$

$$\exists_x[P{:}Q \vee R] \stackrel{val}{=} \exists_x[P{:}Q] \vee \exists_x[P{:}R]$$

Other equivalences with quantifiers

Monotonicity of quantifiers

$$\forall_x[P{:}Q \Rightarrow R] \Rightarrow (\forall_x[P{:}Q] \Rightarrow \forall_x[P{:}R]) \stackrel{val}{=} T$$

$$\forall_x[P{:}Q \Rightarrow R] \Rightarrow (\exists_x[P{:}Q] \Rightarrow \exists_x[P{:}R]) \stackrel{val}{=} T$$

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tautologies

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$$\forall_x[P{:}Q \Rightarrow R] \Rightarrow (\exists_x[P{:}Q] \Rightarrow \exists_x[P{:}R]) \overset{val}{=} T$$

tautologies

Lemma E1: $P \overset{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Other equivalences with quantifiers

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Lemma W5: If $Q \stackrel{val}{\models} R$ then $\forall_x[P{:}Q] \stackrel{val}{\models} \forall_x[P{:}R]$.