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Lemma: The relation $\equiv_n$ is an equivalence for every n.

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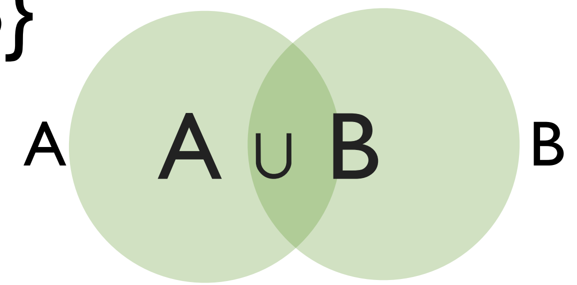
Task: Describe the equivalence classes of $\equiv_n$
How many classes are there?

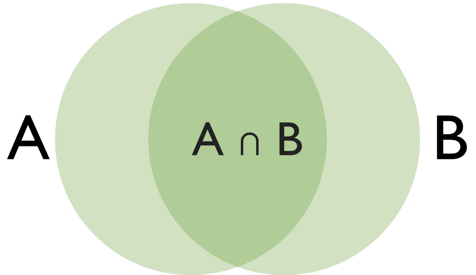
Unions and intersections of multiple sets

Union (Vereinigung) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



Intersection (Durchschnitt) $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

A and B are disjoint if $A \cap B = \emptyset$



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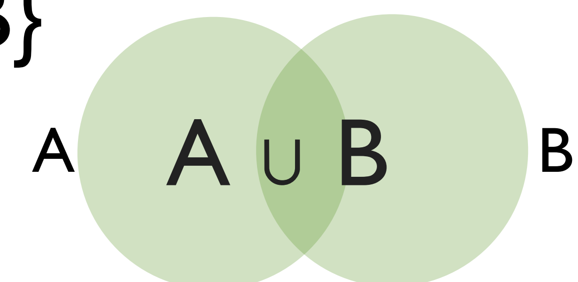

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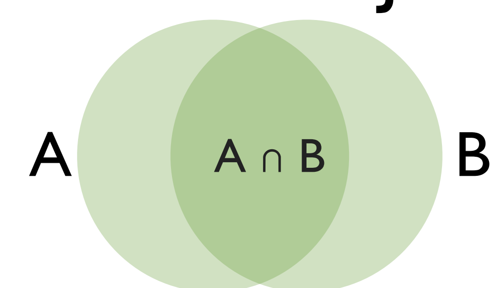

In general, for sets $A_1, A_2, \ldots, A_n$ with $n \geq 1$,

$$A_1 \cup A_2 \cup \ldots \cup A_n = \bigcup_{1 \leq i \leq n} A_i = \{x \mid x \in A_i \text{ for some } i \in \{1,..n\}\}$$

$$A_1 \cap A_2 \cap \ldots \cap A_n = \bigcap_{1 \leq i \leq n} A_i = \{x \mid x \in A_i \text{ for all } i \in \{1,..n\}\}$$

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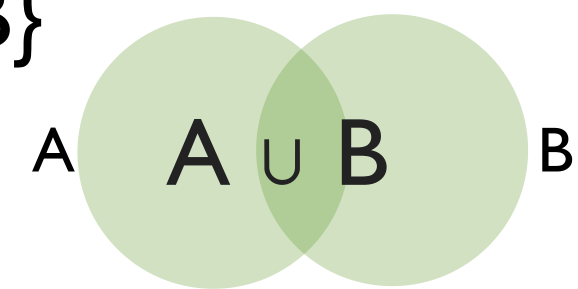

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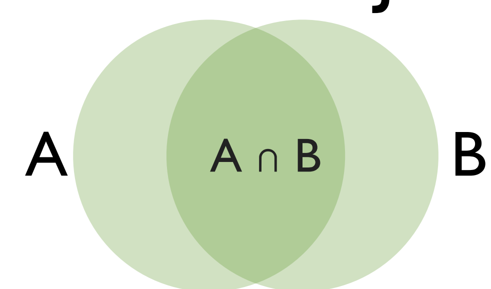

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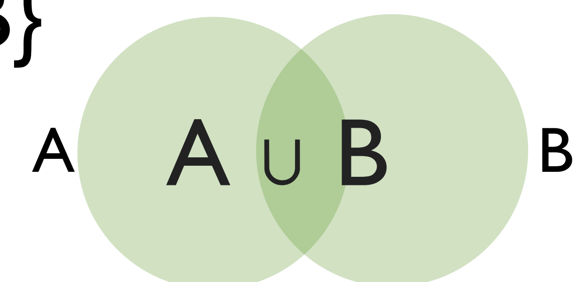

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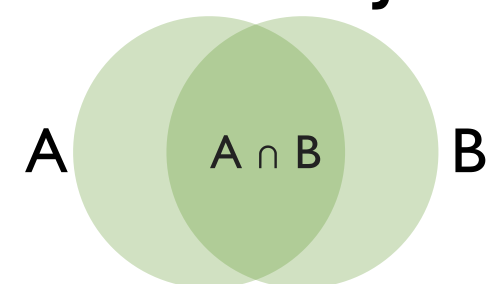

In general, for a family of sets $(A_i \mid i \in I)$

$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$

$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$

Back to equivalence classes

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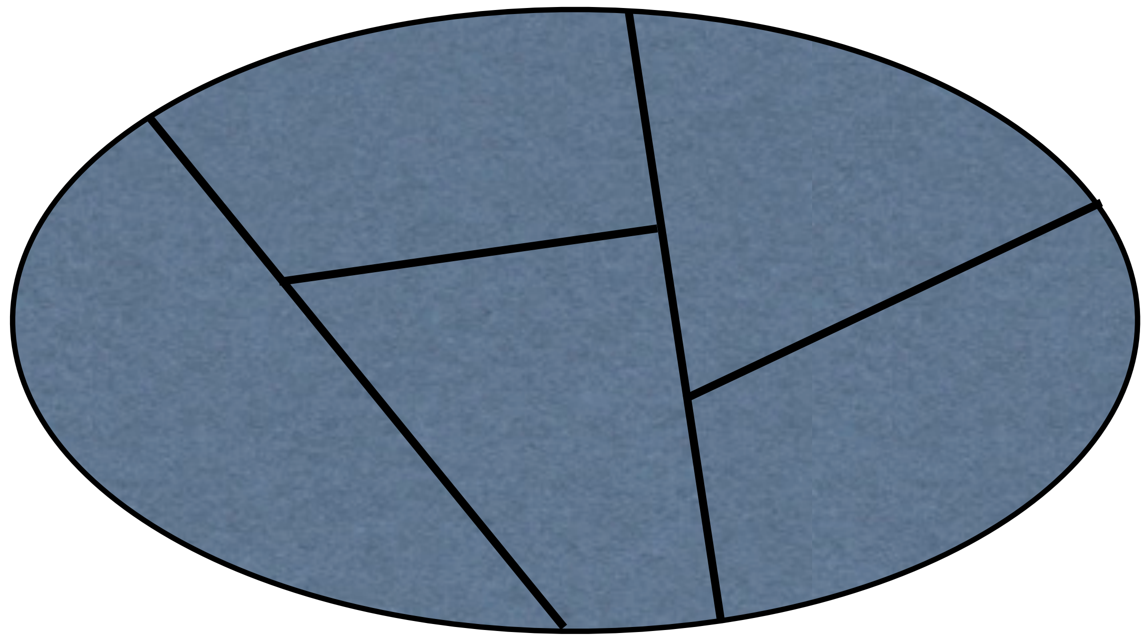
$$([a]_R , a \in A)$$ is a family of sets.

(all equivalence classes of R)

Lemma E2: $A = \bigcup_{a \in A} [a]_R$. The union is disjoint.

Partitions



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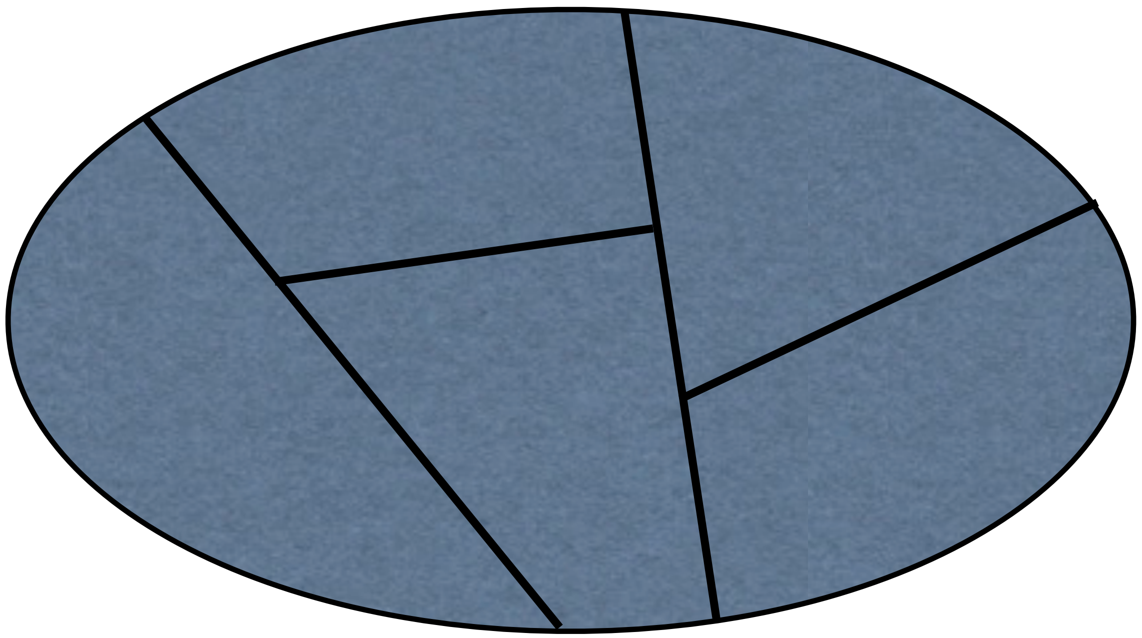

Def. Let X be a set. A subset P of the powerset $\mathcal{P}(X)$ is a partition (Klasseneinteilung) of X if it satisfies:

(1) For all $A \in P$, $A \neq \emptyset$

(2) For all $A, B \in P$, if $A \neq B$ then $A \cap B = \emptyset$

(3) $\bigcup_{A \in P} A = X$

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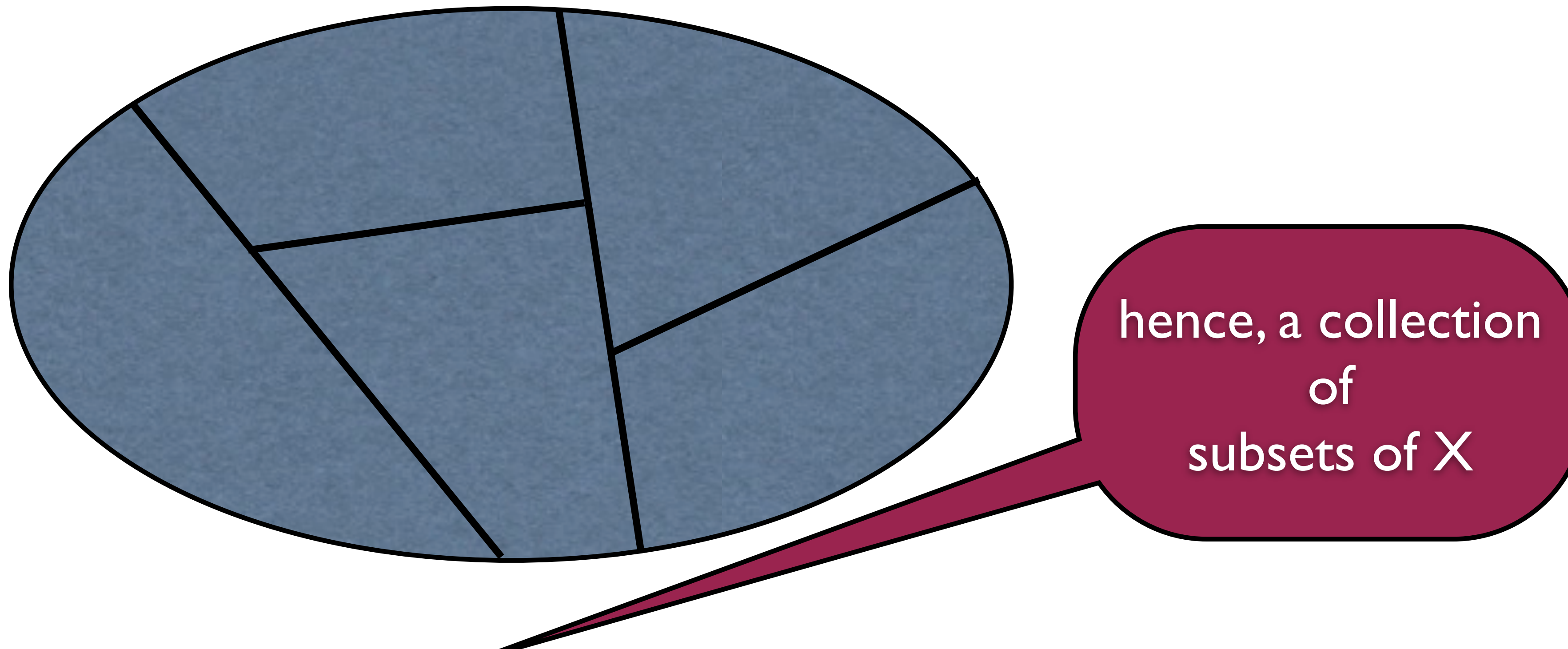

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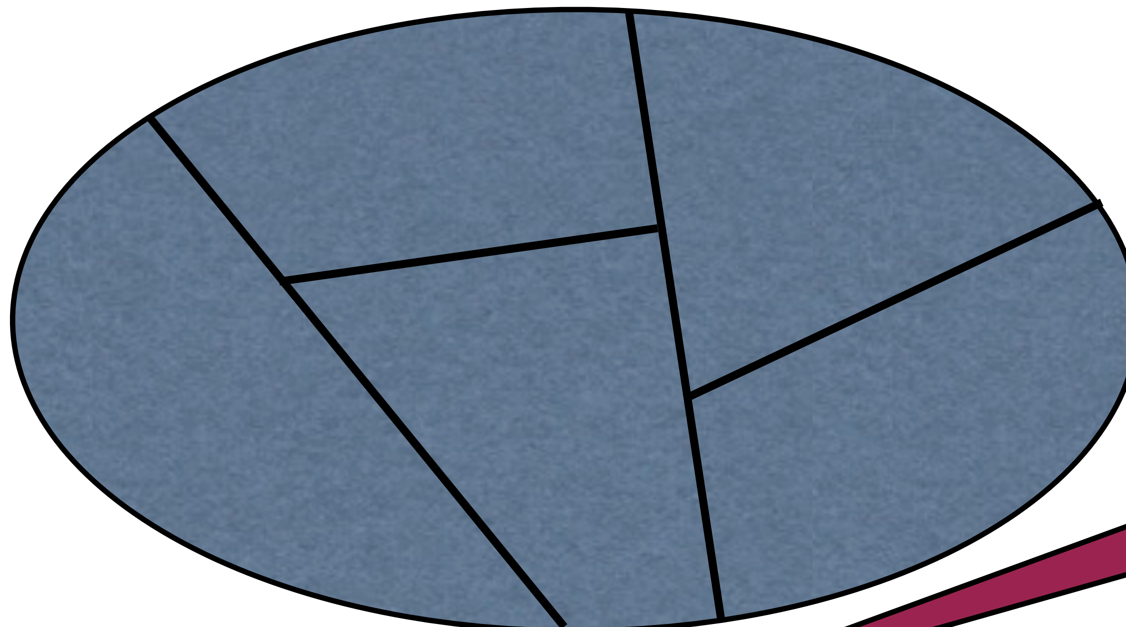

hence, a collection of subsets of X

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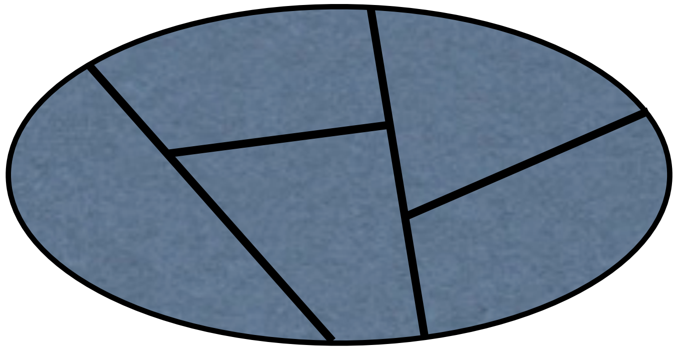
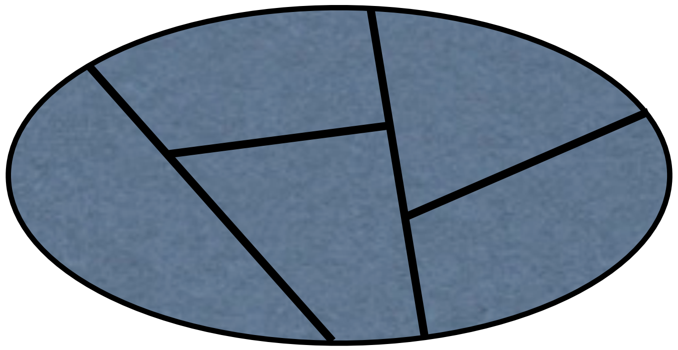
that are non-empty, pairwise disjoint, and their union equals X



Partitions = Equivalences



Partitions = Equivalences

Theorem PE: Let X be a set.

(1) If R is an equivalence on X, then the set

$$P(R) = \{ [x]_R \mid x \in X \}$$

is a partition of X.

(2) If P is a partition of X, then the relation

$$R(P) = \{(x,y) \in X \times X \mid \text{there is } A \in P \text{ such that } x,y \in A\}$$

is an equivalence relation.

Moreover, the assignments $R \mapsto P(R)$ and $P \mapsto R(P)$ are inverse to eachother, i.e., $R(P(R)) = R$ and $P(R(P)) = P$.