Properties of sets

1.	$\varnothing \subseteq X$
2.	If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$
3.	$X \cap Y \subseteq X$, $X \cap Y \subseteq Y$
4.	$X \subseteq X \cup Y$, $Y \subseteq X \cup Y$
5.	If $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$, then $X_1 \cap X_2 \subseteq Y_1 \cap Y_2$
6.	If $X_1 \subseteq Y_1$ and $X_2 \subseteq Y_2$, then $X_1 \cup X_2 \subseteq Y_1 \cup Y_2$
7.	$X \cap Y = X$ iff $X \subseteq Y$
8.	$X \cap X = X$ (idempotence)
9.	$X \cup X = X$ (idempotence)
10.	$X \cap \varnothing = \varnothing$

Properties of sets

11.	$X \cup \varnothing = X$
12.	$X \cap Y = Y \cap X$ (commutativity)
13.	$X \cup Y = Y \cup X$ (commutativity)
14.	$X \cap (Y \cap Z) = (X \cap Y) \cap Z$ (associativity)
15.	$X \cup (Y \cup Z) = (X \cup Y) \cup Z$ (associativity)
16.	$X \cap (X \cup Y) = X$ (absorption)
17.	$X \cup (X \cap Y) = X$ (absorption)
18.	$X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$ (distributivity)
19.	$X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z)$ (distributivity)
20.	$X \setminus Y \subseteq X$

Properties of sets

21.	$(X \setminus Y) \cap Y = \varnothing$
22.	$X \cup Y = X \cup (Y \setminus X)$
23.	$X \setminus X = \varnothing$
24.	$X \setminus \varnothing = X$
25.	$\varnothing \setminus X = \varnothing$
26.	If $X \subseteq Y$, then $X \setminus Y = \varnothing$
27.	$(X^c)^c = X$
28.	$(X \cap Y)^c = X^c \cup Y^c$ (De Morgan)
29.	$(X \cup Y)^c = X^c \cap Y^c$ (De Morgan)
30.	$X \times \varnothing = \varnothing$
31.	$\varnothing \times X = \varnothing$
32.	If $X \subseteq Y$, then $\mathcal{P}(X) \subseteq \mathcal{P}(Y)$

Product of multiple sets

Direct product (Kartesisches Produkt)

$A \times B = \{(x,y) \mid x \in A \text{ and } y \in B\}$

ordered pairs

$(A \times B) \times C \neq A \times (B \times C)$

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In general, for sets $A_1, A_2, \ldots, A_n$ with $n \geq 1$,

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if $A_i = A$ for all i, then the product is denoted A^n

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sequence of length n

Finite sequences, words

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Let A be a set, an aphabet

$A^0 = \{\varepsilon\}$

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sequences (words) of length

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Relations

Def. If A and B are sets, then any subset $R \subseteq A \times B$ is a (binary) relation between A and B

Def. R is a relation on A if $R \subseteq A \times A$

some relations are special

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similarly, unary relation (subset), n-ary relation...

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some relations are special

Special relations

A relation $R \subseteq A \times A$ is:

reflexive	iff	for all $a \in A$, $(a,a) \in R$
symmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \in R$
transitive	iff	for all $a,b,c \in A$, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$
irreflexive	iff	for all $a \in A$, $(a,a) \notin R$
antisymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$
asymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \notin R$
total	iff	for all $a,b \in A$, $(a,b) \in R$ or $(b,a) \in R$

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(infix) notation aRb for $(a,b) \in R$

Special relations

A relation R on A, i.e., $R \subseteq A \times A$ is:

equivalence	iff	R is reflexive, symmetric, and transitive
partial order	iff	R is reflexive, antisymmetric, and transitive
strict order	iff	R is irreflexive and transitive
preorder	iff	R is reflexive and transitive
total (linear) order	iff	R is a total partial order

Obvious properties

1. Every partial order is a preorder.

2. Every total order is a partial order.

3. Every total order is a preorder.

4. If $R \subseteq A \times A$ is a relation that contains cycles,
   i.e. there are $a, b \in A$ such that $a = b$, $(a,b) \in R$ and $(b,a) \in R$,
   then R is not a preorder, nor a partial order, nor a total order.

Operations on relations

Let $R \subseteq A \times B$ and $S \subseteq B \times C$ be two relations. Their composition is the relation

$R \circ S = \{(a,c) \in A \times C \mid \text{there is } b \in B \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\}$

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Let $R \subseteq A \times B$ be a relation. The inverse relation of R is the relation

$$R^{-1} = \{(b,a) \in B \times A \mid (a, b) \in R\}$$

Characterizations

Lemma: Let R be a relation over the set A. Then

1. R is reflexive iff $\Delta_A \subseteq R$
2. R is symmetric iff $R \subseteq R^{-1}$
3. R is transitive iff $R^2 \subseteq R$

Important equivalence on $\mathbb{Z}$

Def. For a natural number n, the relation $\equiv_n$ is defined as

$i \equiv_n j$ iff $n \mid i - j$

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Lemma: The relation $\equiv_n$ is an equivalence for every n.