Derivations / Reasoning with quantifiers

Proving a universal quantification

To prove

$\forall x[x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$

Proof

Let $x \in \mathbb{Z}$ be arbitrary and assume that $x \geq 2$.

Then, for this particular x, it holds that

$$x^2 - 2x = x(x-2) \geq 0 \quad \text{(Why?)}$$

Conclusion: $\forall x[x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$.

∀ introduction

How do we prove a universal quantification?

similar to ⇒-intro

with generating hypothesis

∀-introduction

…

{Assume}

(k) **var** x; P(x)

…

(l-1) Q(x)

{∀-intro on (k) and (l-1)}

(l) ∀x[P(x) : Q(x)]

flag shows the validity of a hypothesis

Using a universal quantification

We know $\forall x[x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$

Whenever we encounter an $a \in \mathbb{Z}$ such that $a \geq 2$, we can conclude that $a^2 - 2a \geq 0$.

For example, $(52387^2 - 2 \cdot 52387) \geq 0$
since $52387 \in \mathbb{Z}$ and $52387 \geq 2$.

∀ elimination

How do we use a universal quantification in a proof?

similar to implication but we need a witness

∀-elimination

|| ||

(k) $\forall x[P(x) : Q(x)]$

|| ||

(l) $P(a)$

a is an object (variable, number,..) which is "known" in line (l)

|| ||

{∀-elim on (k) and (l)}

(m) $Q(a)$

the same "a" from line (l)

($k < m, l < m$)

time for an example!

∃ introduction

How do we prove an existential quantification?

$\neg \forall x[P(x):\neg Q(x)] \overset{val}{\models} \exists x\, [P(x) : Q(x)]$

∃-introduction

…

{Assume}

(k) $\boxed{\forall x[P(x) : \neg Q(x)]}$

…

(l-1) F

{∃-intro on (k) and (l-1)}

(l) $\exists x\, [P(x) : Q(x)]$

and ¬-intro

∃ elimination

How do we use an existential quantification in a proof?

$\exists x\,[P(x) : Q(x)] \overset{val}{\models} \neg\, \forall x[P(x){:}\neg Q(x)]$

∃-elimination

	⋮
(k)	$\exists x\,[P(x) : Q(x)]$
	⋮
(l)	$\forall x[P(x){:}\ \neg Q(x)]$
	⋮
	{∃-elim on (k) and (l)}
(m)	F

(k < m, l < m)

and ¬-elimination

time for an example!

Proofs with ∃-introduction and ∃-elimination are unnecessarily long and cumbersome…

There are alternatives!

Proving an existential quantification

To prove

$\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$

Proof

It suffices to find a witness, i.e., an $x \in \mathbb{Z}$ satisfying

$$x^3 - 2x - 8 \geq 0.$$

$x = 3$ is a witness, since $3 \in \mathbb{Z}$ and $3^3 - 2 \cdot 3 - 8 = 13 \geq 0$

also $x = 5$ is a witness…

Conclusion: $\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$.

Alternative ∃ introduction

How do we prove an existential quantification?

by finding
a witness

∃

…

(k) P(a)

…

(l) Q(a)

…

{∃*-intro on (k) and (l)}

(m) ∃x [P(x) : Q(x)]

strategy: wait until a witness object appears

does not always work

(k < m, l < m)

Using an existential quantification

We know $\exists x[x \in \mathbb{R} : a - x < 0 < b - x]$

We can declare an $x \in \mathbb{Z}$ (a witness) such that

$$a - x < 0 < b - x$$

and use it further in the proof. For example:

From $a - x < 0$, we get $a < x$.
From $b - x > 0$, we get $x < b$.
Hence, $a < b$.

Alternative ∃ elimination

How do we use an existential quantification in a proof?

we pick a witness

∃

|| ||

(k) ∃x [P(x) : Q(x)]

|| ||

{∃*-elim on (k)}
(m) Pick x with P(x) and Q(x)

x must be new!

(k < m)

time for an example!