Formale Systeme

Week 6

Equivalences with quantifiers

http://cs.uni-salzburg.at/~anas/teaching/FormaleSysteme/

Equivalences with quantifiers

Bound variables

$$\forall_x[P : Q] \stackrel{val}{=} \forall_y[P[y \text{ for } x] : Q[y \text{ for } x]]$$

$$\exists_x[P : Q] \stackrel{val}{=} \exists_y[P[y \text{ for } x] : Q[y \text{ for } x]]$$

if y is not free in P and Q

Domain splitting

Examples:

Domain splitting

Examples:

$$\forall_x[x \leqslant 1 \vee x \geqslant 5 : x^2 - 6x + 5 \geqslant 0]$$

$$\stackrel{val}{=} \quad \forall_x[x \leqslant 1 : x^2 - 6x + 5 \geqslant 0] \wedge \forall_x[x \geqslant 5 : x^2 - 6x + 5 \geqslant 0]$$

Domain splitting

Examples:

$$\forall_x[x \leqslant 1 \lor x \geqslant 5 : x^2 - 6x + 5 \geqslant 0]$$
$$\stackrel{val}{=} \forall_x[x \leqslant 1 : x^2 - 6x + 5 \geqslant 0] \land \forall_x[x \geqslant 5 : x^2 - 6x + 5 \geqslant 0]$$

$$\exists_k[0 \leqslant k \leqslant n : k^2 \leqslant 10]$$
$$\stackrel{val}{=} \exists_k[0 \leqslant k \leqslant n-1 \lor k = n : k^2 \leqslant 10]$$
$$\stackrel{val}{=} \exists_k[0 \leqslant k \leqslant n-1 : k^2 \leqslant 10] \lor \exists_k[k = n : k^2 \leqslant 10]$$

Domain splitting

Examples:

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$$\stackrel{val}{=} \forall_x[x \leqslant 1 : x^2 - 6x + 5 \geqslant 0] \wedge \forall_x[x \geqslant 5 : x^2 - 6x + 5 \geqslant 0]$$

$$\exists_k[0 \leqslant k \leqslant n : k^2 \leqslant 10]$$
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$$\stackrel{val}{=} \exists_k[0 \leqslant k \leqslant n-1 : k^2 \leqslant 10] \vee \exists_k[k = n : k^2 \leqslant 10]$$

Domain splitting

$$\forall_x[P \vee Q : R] \stackrel{val}{=} \forall_x[P : R] \wedge \forall_x[Q : R]$$
$$\exists_x[P \vee Q : R] \stackrel{val}{=} \exists_x[P : R] \vee \exists_x[Q : R]$$

Equivalences with quantifiers

One-element rule

$$\forall_x[x = n : Q] \stackrel{val}{=} Q[n \text{ for } x]$$

$$\exists_x[x = n : Q] \stackrel{val}{=} Q[n \text{ for } x]$$

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Example:

$$\forall_x[x = 3 : 2 \cdot x \geqslant 1] \stackrel{val}{=} 2 \cdot 3 \geqslant 1$$

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Empty domain

$$\forall_x[F : Q] \stackrel{val}{=} T$$

$$\exists_x[F : Q] \stackrel{val}{=} F$$

Equivalences with quantifiers

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Empty domain

$$\forall_x[F : Q] \stackrel{val}{=} T$$

$$\exists_x[F : Q] \stackrel{val}{=} F$$

"All Marsians are green"

Domain weakening

Intuition: The following are equivalent

$$\forall_x[x \in D : A(x)] \quad \text{and} \quad \forall_x[x \in D \Rightarrow A(x)]$$
$$\exists_x[x \in D : A(x)] \quad \text{and} \quad \exists_x[x \in D \wedge A(x)]$$

The same can be done to parts of the domain

Domain weakening

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Domain weakening

$$\forall_x[P \wedge Q : R] \overset{val}{=} \forall_x[P : Q \Rightarrow R]$$
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Domain weakening

$$\forall_x[P \wedge Q : R] \stackrel{val}{=} \forall_x[P : Q \Rightarrow R]$$
$$\exists_x[P \wedge Q : R] \stackrel{val}{=} \exists_x[P : Q \wedge R]$$

$$P \wedge Q \stackrel{val}{\models} P$$

Equivalences with quantifiers

De Morgan

$$\neg\forall_x[P:Q] \stackrel{val}{=} \exists_x[P:\neg Q]$$
$$\neg\exists_x[P:Q] \stackrel{val}{=} \forall_x[P:\neg Q]$$

Equivalences with quantifiers

De Morgan

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not for all = at least for one not

not exists = for all not

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Hence: $\neg\forall = \exists\neg$ and $\neg\exists = \forall\neg$

Equivalences with quantifiers

De Morgan

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not for all = at least for one not

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Hence: $\neg\forall = \exists\neg$ and $\neg\exists = \forall\neg$

It holds further that:

$$\neg\forall_x\neg = \exists_x\neg\neg = \exists_x$$
$$\neg\exists_x\neg = \forall_x\neg\neg = \forall_x$$