Solutions to selected exercises of Chapters 1–6

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This document contains solutions to the following exercises in the book [1]:

2.4(c),(d), 2.6, 2.8, 3.1, 4.4(e), (f), 5.5 (c), (d), 6.5(b) and 6.6(c).

We **strongly** advise you to first try all these exercises by yourself, before looking at all at the solutions below. There is not a lot of variation possible in the way solutions to exercises should be written down. So if your solution in one way or another deviates from a solution below, then consider discussing the differences with your instructor.

2.4 (c) We show how $((\neg(\neg a)) \Rightarrow ((\neg a) \wedge b))$ is built according to Definition 2.3.1:

(i) a and b are propositional variables, so according to *Basis* they are abstract propositions;

(ii) since a is an abstract proposition, by *Step (case1)* so is $(\neg a)$;

(iii) since $(\neg a)$ is an abstract proposition, by *Step (case 1)* so is $(\neg(\neg a))$;

(iv) since $(\neg a)$ and b are abstract propositions, by *Step (case 2)* so is $((\neg a) \wedge b)$;

(v) since $(\neg(\neg a))$ and $((\neg a) \wedge b)$ are abstract propositions, so is $((\neg(\neg a)) \Rightarrow ((\neg a) \wedge b))$.

Alternatively, the reasoning above may be written a bit more concisely in the form of a *proof tree* (we use B to abbreviate *Basis*, *S1* to abbreviate *Step (case 1)*, and *S2* to abbreviate *Step (case 2)*):

$$\dfrac{\dfrac{\dfrac{\dfrac{}{a}\,B}{(\neg a)}\,S1}{(\neg(\neg a))}\,S1 \qquad \dfrac{\dfrac{\dfrac{}{a}\,B}{(\neg a)}\,S1 \quad \dfrac{}{b}\,B}{((\neg a) \wedge b)}\,S2}{((\neg(\neg a)) \Rightarrow ((\neg a) \wedge b))}\,S2$$

(d) We show how $(a \Rightarrow ((b \wedge a) \vee c))$ is built according to Definition 2.3.1:

(i) a, b and c are propositional variables, so according to *Basis* they are abstract propositions;

(ii) since b and a are abstract propositions, by *Step (case 2)* so is $(b \wedge a)$;

(iii) since $(b \wedge a)$ and c are abstract propositions, by *Step (case 2)* so is $((b \wedge a) \vee c)$;

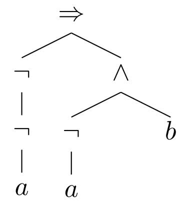
(iv) since a and $((b \wedge a) \vee c)$ are abstract propositions, by *Step (case 2)* so is $(a \Rightarrow ((b \wedge a) \vee c)$.

Again, we may alternatively write the above reasoning in the form of a *proof tree* (using the same abbreviations B, *S1* and *S2*):

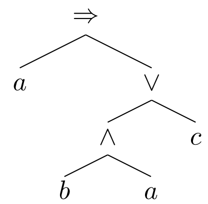
$$\mathit{S2}\,\dfrac{B\,\dfrac{}{a} \qquad \mathit{S2}\,\dfrac{\mathit{S2}\,\dfrac{B\,\dfrac{}{b} \quad B\,\dfrac{}{a}}{(b \wedge a)} \quad B\,\dfrac{}{c}}{((b \wedge a) \vee c)}}{(a \Rightarrow ((b \wedge a) \vee c))}$$

2.6 [only for (c) and (d) of Exercise 2.4]

(a) The tree for the abstract proposition of Exercise 2.4(c) is:



The tree for the abstract proposition of Exercise 2.4(d) is:



(b) The main symbol of the abstract proposition of Exercise 2.4(c) is $\Rightarrow$.
The main symbol of the abstract proposition of Exercise 2.4(d) is $\Rightarrow$.

2.8 [only for (c) and (d) of Exercise 2.4]

First, consider the abstract proposition

$$((\neg(\neg a)) \Rightarrow ((\neg a) \wedge b)) \ .$$

of Exercise 2.4(c). We may always drop the outermost parentheses:

$$(\neg(\neg a)) \Rightarrow ((\neg a) \wedge b) \ .$$

Since $\neg$ has higher priority than $\Rightarrow$ and $\wedge$ according to the priority scheme on p. 13 of the book [1], we may drop the parentheses around $(\neg(\neg a))$ in $(\neg(\neg a)) \Rightarrow ((\neg a) \wedge b)$ and around $(\neg a)$ in $((\neg a) \wedge b)$. We get

$$\neg(\neg a) \Rightarrow (\neg a \wedge b) \ .$$

Since $\wedge$ has a higher priority than $\Rightarrow$ according to the priority scheme on p. 13 of the book [1], we may drop the parentheses around $(\neg a \wedge b)$ in $\neg(\neg a) \Rightarrow (\neg a \wedge b)$:

$$\neg(\neg a) \Rightarrow \neg a \wedge b \ .$$

Finally, note that no ambiguity is introduced if we omit the parentheses around the negation $(\neg a)$ in $\neg(\neg a)$, so it is safe to omit them:

$$\neg\neg a \Rightarrow \neg a \wedge b \ .$$

NB: Strictly speaking, the book [1] does not give a rule for this last step. Note, however, that, since $\neg$ is unary, writing $\neg\neg P$ for $\neg(\neg P)$ will never cause ambiguity.

Next, consider the abstract proposition

$$(a \Rightarrow ((b \wedge a) \vee c)) \ .$$

of Exercise 2.4(d). We may, as always, omit the outermost parentheses, to get:

$$a \Rightarrow ((b \wedge a) \vee c) \ .$$

Then, since $\vee$ has a higher priority than $\Rightarrow$, we may also omit the parentheses around $((b \wedge a) \vee c$ in $a \Rightarrow ((b \wedge a) \vee c)$ to get

$$a \Rightarrow (b \wedge a) \vee c \ .$$

NB: We do *not* agree with the suggestion in the exercise to use the left-associativity rule, because it goes against Convention 2.5.2 on p. 14 of the book [1], where it is stated explicitly that the rule for left-associativity is not going to be used. (If we would have applied the left-associativity for $\wedge$ and $\vee$, as the exercise suggests, then we could also have omitted the last pair of parentheses in the above abstract proposition.)

3.1 [only for (c) and (d) of Exercise 2.4]

The truth table of the abstract proposition of Exercise 2.4(c) is:

a	b	$\neg a$	$\neg\neg a$	$\neg a \wedge b$	$\neg\neg a \Rightarrow \neg a \wedge b$
0	0	1	0	0	1
0	1	1	0	1	1
1	0	0	1	0	0
1	1	0	1	0	0

The truth table of the abstract proposition of Exercise 2.4(d) is:

a	b	c	$b \wedge a$	$(b \wedge a) \vee c$	$a \Rightarrow (b \wedge a) \vee c$
0	0	0	0	0	1
0	0	1	0	1	1
0	1	0	0	0	1
0	1	1	0	1	1
1	0	0	0	0	0
1	0	1	0	1	1
1	1	0	1	1	1
1	1	1	1	1	1

4.4 (e) To show that $a \Rightarrow (b \Rightarrow a)$ and $a \Rightarrow a$ are equivalent, we first construct a *combined truth table* for these two abstract propositions:

a	b	$b \Rightarrow a$	$a \Rightarrow (b \Rightarrow a)$	$a \Rightarrow a$
0	0	1	1	1
0	1	0	1	1
1	0	1	1	1
1	1	1	1	1

Now, since the columns for $a \Rightarrow (b \Rightarrow a)$ and $a \Rightarrow a$ in the combined truth table are identical, it follows that $a \Rightarrow (b \Rightarrow a)$ and $a \Rightarrow a$ are indeed equivalent.

(f) To show that $(a \wedge b) \vee b$ and $(b \wedge c) \vee (b \wedge \neg c)$ are equivalent, we first construct a *combined truth table* for these two abstract propositions:

a	b	c	$a \wedge b$	$(a \wedge b) \vee b$	$b \wedge c$	$\neg c$	$b \wedge \neg c$	$(b \wedge c) \vee (b \wedge \neg c)$
0	0	0	0	0	0	1	0	0
0	0	1	0	0	0	0	0	0
0	1	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
1	0	0	0	0	0	1	0	0
1	0	1	0	0	0	0	0	0
1	1	0	1	1	0	1	1	1
1	1	1	1	1	1	0	0	1

Now, since the columns for $(a \wedge b) \vee b$ and $(b \wedge c) \vee (b \wedge \neg c)$ in the combined truth table are identical, it follows that $a \Rightarrow (b \Rightarrow a)$ and $a \Rightarrow a$ are indeed equivalent.

5.5 (c) To show that disjunction ($\vee$) distributes over bi-implication ($\Leftrightarrow$) we need prove the following equivalence:

$$P \vee (Q \Leftrightarrow R) \stackrel{val}{==} (P \vee Q) \Leftrightarrow (P \vee R) \ .$$

To this end, we construct a combined truth table for both sides of the equivalence:

P	Q	R	$Q \Leftrightarrow R$	$P \vee (Q \Leftrightarrow R)$	$P \vee Q$	$P \vee R$	$(P \vee Q) \Leftrightarrow (P \vee R)$
0	0	0	1	1	0	0	1
0	0	1	0	0	0	1	0
0	1	0	0	0	1	0	0
0	1	1	1	1	1	1	1
1	0	0	1	1	1	1	1
1	0	1	0	1	1	1	1
1	1	0	0	1	1	1	1
1	1	1	1	1	1	1	1

Since the columns for $P \vee (Q \Leftrightarrow R)$ and $(P \vee Q) \Leftrightarrow (P \vee R)$ in the combined truth table are identical, it follows that they are equivalent.

(d) To prove that the equivalence

$$P \wedge (Q \Leftrightarrow R) \stackrel{val}{==} (P \wedge Q) \Leftrightarrow (P \wedge R)$$

does *not* hold, it suffices to give a counterexample: Take $P = 0$, $Q = 0$ and $R = 0$. Then the left-hand side of the equivalence evaluates to 0, while the right-hand side of the equivalence evaluates to 1. Hence, the equivalence does not hold.

6.5 (b) To prove that $((Q \Rightarrow P) \Rightarrow \neg Q) \Leftrightarrow (\neg P \vee \neg Q)$ is a tautology, by Lemma 6.1.3 it suffices to establish the equivalence

$$(Q \Rightarrow P) \Rightarrow \neg Q \stackrel{val}{==} \neg P \vee \neg Q \ .$$

We establish the equivalence with the following calculation:

$$
\begin{array}{ll}
 & (Q \Rightarrow P) \Rightarrow \neg Q \\
\stackrel{val}{==} & \{ \text{ Implication } \} \\
 & \neg(Q \Rightarrow P) \vee \neg Q \\
\stackrel{val}{==} & \{ \text{ Implication } \} \\
 & \neg(\neg Q \vee P) \vee \neg Q \\
\stackrel{val}{==} & \{ \text{ De Morgan } \} \\
 & (\neg\neg Q \wedge \neg P) \vee \neg Q \\
\stackrel{val}{==} & \{ \text{ Double Negation } \} \\
 & (Q \wedge \neg P) \vee \neg Q \\
\stackrel{val}{==} & \{ \text{ Distributivity } \} \\
 & (Q \vee \neg Q) \wedge (\neg P \vee \neg Q) \\
\stackrel{val}{==} & \{ \text{ Excluded Middle } \} \\
 & \texttt{True} \wedge (\neg P \vee \neg Q) \\
\stackrel{val}{==} & \{ \texttt{ True}/\texttt{False}\text{-elimination } \} \\
 & \neg P \vee \neg Q
\end{array}
$$

6.6 (c) To prove that $((P \wedge \neg R) \vee (\neg P \wedge R)) \Leftrightarrow (P \Leftrightarrow \neg R)$ is a tautology, by Lemma 6.1.3 it suffices to establish the equivalence

$$(P \wedge \neg R) \vee (\neg P \wedge R) \stackrel{val}{==} P \Leftrightarrow \neg R$$

We establish the equivalence with the following calculation[1]:

$$
\begin{array}{ll}
 & P \Leftrightarrow \neg R \\
\stackrel{val}{==} & \{ \text{ Bi-implication } \} \\
 & (P \Rightarrow \neg R) \wedge (\neg R \Rightarrow P) \\
\stackrel{val}{==} & \{ \text{ Implication } (2\times) \} \\
 & (\neg P \vee \neg R) \wedge (\neg\neg R \vee P) \\
\stackrel{val}{==} & \{ \text{ Double Negation } \} \\
 & (\neg P \vee \neg R) \wedge (R \vee P) \\
\stackrel{val}{==} & \{ \text{ Distributivity } \} \\
 & (\neg P \wedge (R \vee P)) \vee (\neg R \wedge (R \vee P)) \\
\stackrel{val}{==} & \{ \text{ Distributivity } (2\times) \} \\
 & ((\neg P \wedge R) \vee (\neg P \wedge P)) \vee ((\neg R \wedge R) \vee (\neg R \wedge P)) \\
\stackrel{val}{==} & \{ \text{ Contradiction } (2\times) \} \\
 & ((\neg P \wedge R) \vee \texttt{False}) \vee (\texttt{False} \vee (\neg R \wedge P)) \\
\stackrel{val}{==} & \{ \texttt{ True}/\texttt{False}\text{-elimination } \} \\
 & (P \wedge \neg R) \vee (\neg P \wedge R)
\end{array}
$$

[1] We find it, in this case, convenient to start with the right-hand side of the equation; note that this is allowed by Lemma 6.1.1(2).

References

[1] Rob Nederpelt and Fairouz Kamareddine. *Logical Reasoning: A First Course*, volume 3 of *Texts in Computing*. King's College Publications, second revised edition edition, 2011.