Formale Systeme

Example tasks for partial exam 2

Task 1. (20 points) Prove with a derivation that the following formula is a tautology

$$\exists_x \forall_y [P(x) \Rightarrow Q(y)] \Rightarrow (\forall_u [P(u)] \Rightarrow \exists_v [Q(v)])$$

Task 2. (20 points) Check whether the following propositions always hold. If so, give a proof; if not, give a counterexample.

(a) $\mathcal{P}(A) \cap \mathcal{P}(B) = \mathcal{P}(A \cap B)$

(b) $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$

Task 3. (20 points) Prove that a relation R is transitive if and only if $R^2 \subseteq R$. Use this then to show that the transitive closure of a transitive relation R is R itself.

Task 4. (20 points) Let $f\colon A \to B$ be given. Show that

(a) f is injective if and only if for any two functions $g_1, g_2\colon C \to A$ it holds that $f \circ g_1 = f \circ g_2 \implies g_1 = g_2$.

(b) f is surjective if and only if for any two functions $g_1, g_2\colon B \to C$ it holds that $g_1 \circ f = g_2 \circ f \implies g_1 = g_2$.

Task 5. (20 points) For two sets A and B, by A^B we denote the set of all mappings from B to A, i.e., $A^B = \{f \mid f\colon B \to A\}$. What is the cardinality of A^B if $A = \{0, 1, 2, 3, 4\}$ and $B = \{a, b, c\}$? Argument your answer. Prove that $\{0, 1\}^{\mathbb{N}}$ is nondenumerable (uncountable).