Coalgebra for Computer Scientists

www.cs.uni-salzburg/~anas/teaching/Coalgebra

Lecturer: Ana Sokolova
University of Salzburg

TU Vienna, 15.3.2012

We started last week

- with an informal introduction
- coalgebras $S \to \boxed{\ldots \mathrm{S} \ldots}$
- and discussed a bit where do such structures appear in computer science

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state space

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Sequences

A very concrete coalgebra

$$\texttt{next}: A^{\infty} \longrightarrow \{\bot\} \cup A \times A^{\infty}$$

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type $\{\bot\} \cup A \times (-)$

$$\mathtt{next}(\sigma) = \begin{cases} \bot & \text{if } \sigma = \varepsilon \\ (a, \sigma') & \text{if } \sigma = a \cdot \sigma',\ a \in A,\ \sigma' \in A^\infty \end{cases}$$

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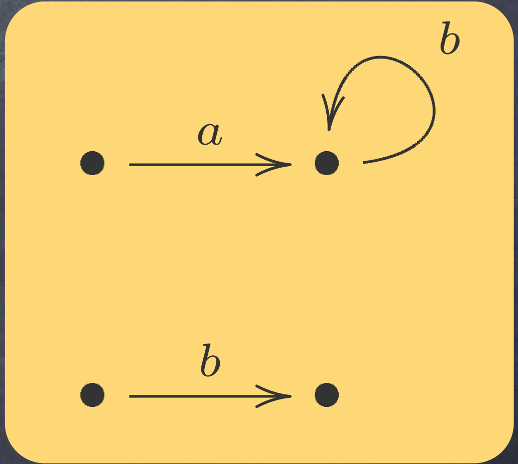
Example:



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Proposition The coalgebra `next` is final.

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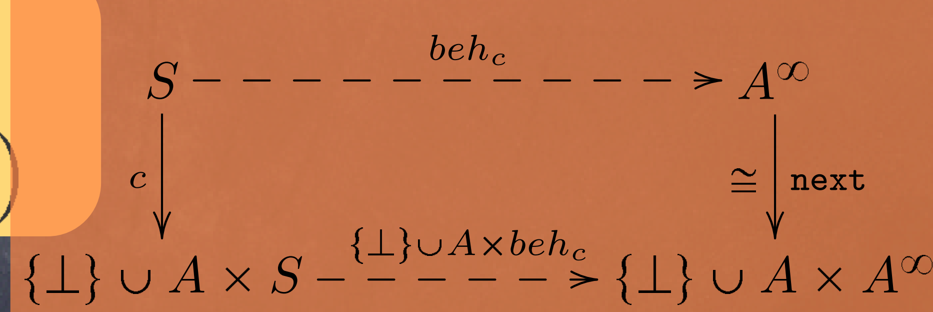

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Coinduction definition principle

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$$\begin{array}{ccc} S & \xrightarrow{beh_c} & A^\infty \\ {\scriptstyle c}\downarrow & & \downarrow{\scriptstyle \cong\ \texttt{next}} \\ \{\bot\} \cup A \times S & \xrightarrow{\{\bot\} \cup A \times beh_c} & \{\bot\} \cup A \times A^\infty \end{array}$$

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Coinduction proof principle

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