Turing Machines

= finite control (automaton states)
+ (potentially) infinite tape
+ head for reading/writing

unlimited access
to infinite memory!

Turing Machine

Definition

A Turing machine M is a tuple $M = (Q, \Sigma, \Gamma, \delta, q_0, !, q_{rej})$ where

Q is a finite set of states
Σ is the input alphabet ($\square \notin \Sigma$)
Γ is the tape alphabet ($\Sigma \subseteq \Gamma$, $\square \in \Gamma$)
$\delta: Q \setminus \{!, q_{rej}\} \times \Gamma \longrightarrow Q \times \Gamma \times \{L,R\}$ is the transition function

or {L,R,N}

no transitions from ! and q_{rej}

q_0 is the initial state, $q_0 \in Q$
! (or q_{acc}) is the accept state, $! \in Q$
q_{rej} is the reject state, $q_{rej} \in Q$

$\delta(q,a) = (r,b,X)$ means that in a state q, reading the symbol a from the cell on which the head is positioned, the TM changes to state r, writes c in place of b, and moves the head for one cell in direction X

Turing machines

Compute via configurations

(q,w,i)
shortly uqv
if w = uv, |u| = i-1

Given $M = (Q, \Sigma, \Gamma, \delta, q_0, !, q_{rej})$ a configuration of M is an element in

$Q \times \Gamma^* \times \mathbb{N}$ (current state, input word, position of the head)

an initial configuration is $(q_0, w, 1)$.

one-step computation

$uaqbv \models uracv$ iff $\delta(q,b) = (r,c,L)$

$uaqbv \models uacrv$ iff $\delta(q,b) = (r,c,R)$

$qbv \models rcv$ iff $\delta(q,b) = (r,c,L)$

$uaq \models uacr$ iff $\delta(q,\square) = (r,c,R)$

Definition (Turing recognisable language)

The language recognised / accepted by a Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, !, q_{rej})$ is

zero-or-more-steps computation

$$L(M) = \{w \in \Sigma^* \mid (q_0, w, 1) \models^* (!, u, j) \text{ for some } u \text{ and } j\}$$

Turing machines

On input w may

(1) accept w, if $(q_0, w, 1) \vDash^* (!, u, j)$ for some u and j
(2) reject w, if $(q_0, w, 1) \vDash^* (q_{rej}, u, j)$ for some u and j
(3) loop, i.e., never reach ! or q_{rej}

both (2) and (3) are not accepting

A decider TM on input w may only

(1) accept w, if $(q_0, w, 1) \vDash^* (!, u, j)$ for some u and j
(2) reject w, if $(q_0, w, 1) \vDash^* (q_{rej}, u, j)$ for some u and j

deciders do not loop

Definition (Turing decidable language)

A language L is Turing decidable if L = L(M) for some Turing machine $M = (Q, \Sigma, \Gamma, \delta, q_0, !, q_{rej})$ which is a decider.