The Microcosm Principle and Concurrency in Coalgebra

Ana Sokolova
University of Salzburg, Austria

Ichiro Hasuo
Kyoto University, Japan
PRESTO Promotion Program, Japan

Bart Jacobs
Radboud University Nijmegen, NL
Technical University Eindhoven, NL

A short review of coalgebra/coinduction

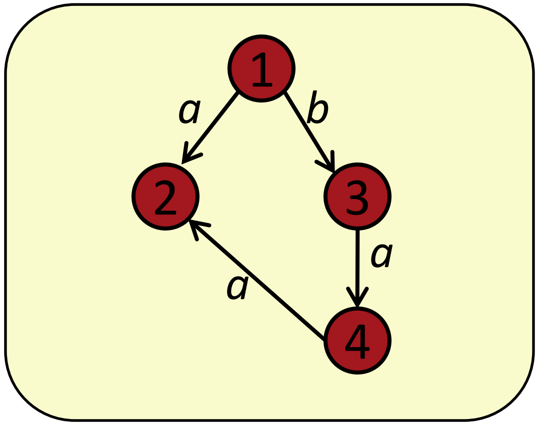
Theory of **coalgebra**

C… …ical theory of state-based systems

in **Sets** : bisimilarity
in Kleisli: trace semantics
[Hasuo,Jacobs,Sokolova LMCS´07]

	…ategorically	
…n	coalgebra	$\begin{array}{c} FX \\ \uparrow \\ X \end{array}$
…vior- p…erving …nap	morphism of coalgebras	$\begin{array}{ccc} FX & \xrightarrow{Ff} & FY \\ \uparrow & & \uparrow \\ X & \xrightarrow[f]{} & Y \end{array}$
behavior	**coinduction** (via final coalgebra)	$\begin{array}{ccc} FX & \dashrightarrow & FZ \\ c\uparrow & & \cong\uparrow \text{final} \\ X & \underset{\mathbf{beh}(c)}{\dashrightarrow} & Z \end{array}$

Coalgebra example – LTS



C = Sets, $F = P_{fin}(\Sigma \times \_)$

F-coalgebra = LTS

coalgebra $c: X \to FX$

states $X = \{1,2,3,4\}$ labels $\Sigma = \{a,b\}$

transitions $c(1) = \{(a,2), (b,3)\}$, $c(2) = \varnothing$, ...

Concurrency

$C \,||\, D$

running C and D in parallel

is **everywhere**

- computer networks
- multi-core processors
- modular, component-based design of complex systems

is **hard to get right**

- exponentially growing complexity
- need for a compositional verification

Compositionality

aids **compositional verification**

Behavior of *C* || *D*
is determined by
behavior of *C* and behavior of *D*

Conventional presentation

$$\mathcal{C}_1 \sim \mathcal{C}_2 \quad \text{and} \quad \mathcal{D}_1 \sim \mathcal{D}_2 \quad \Longrightarrow \quad \mathcal{C}_1 \parallel \mathcal{D}_1 \sim \mathcal{C}_2 \parallel \mathcal{D}_2$$

behavioral equivalence

- bisimilarity
- trace equivalence
- …

„bisimilarity is a congruence“

Compositionality in coalgebra

$\|$: $\mathrm{Coalg}_F \times \mathrm{Coalg}_F \rightarrow \mathrm{Coalg}_F$

composing coalgebras/systems

FZ

$\cong\uparrow$ final

Z

beh(c)

aic compositionality

$$\mathbf{beh}\left(\begin{smallmatrix} FX \\ c\uparrow \\ X \end{smallmatrix} \;\|\; \begin{smallmatrix} FY \\ d\uparrow \\ Y \end{smallmatrix}\right) = \mathbf{beh}\left(\begin{smallmatrix} FX \\ c\uparrow \\ X \end{smallmatrix}\right) \;\|\; \mathbf{beh}\left(\begin{smallmatrix} FY \\ d\uparrow \\ Y \end{smallmatrix}\right)$$

$\|$: $Z \times Z \rightarrow Z$

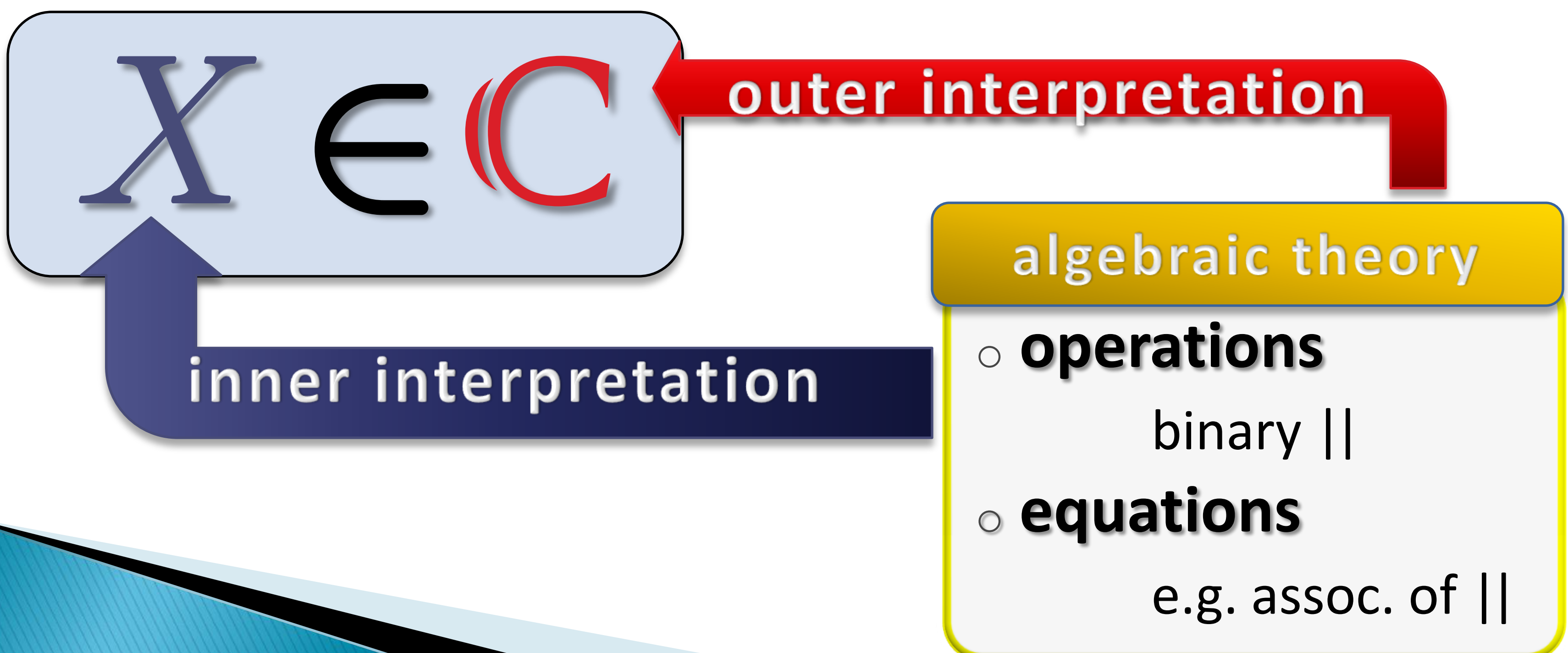
composing behavior

Nested algebraic structures: *the microcosm principle*

$$\begin{array}{ccccc} \mathrm{Coalg}_F & \times & \mathrm{Coalg}_F & \xrightarrow{\;\|\;} & \mathrm{Coalg}_F \\ Z & \times & Z & \xrightarrow{\;\|\;} & Z \end{array}$$

with

$$\begin{pmatrix} FZ \\ \cong \uparrow \text{final} \\ Z \end{pmatrix} \in \mathrm{Coalg}_F$$



Microcosm in macrocosm

We name this principle the *microcosm principle, after the theory, common in pre-modern correlative cosmologies,* that every feature of the microcosm (e.g. the human soul) corresponds to some feature of the macrocosm.

John Baez & James Dolan

Higher-Dimensional Algebra III:

n-Categories and the Algebra of Opetopes

Adv. Math. 1998

The microcosm principle: you may have seen it

monoid in a monoidal category

monoidal cat. $\mathbb{C}$		monoid $M \in \mathbb{C}$
$\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ $I \in \mathbb{C}$	mult. unit	$M \otimes M \xrightarrow{m} M$ $I \xrightarrow{e} M$
$I \otimes X \cong X \cong X \otimes I$	unit law	$\begin{array}{ccccc} M & \longrightarrow & M \otimes M & \longleftarrow & M \\ & = & \downarrow & = & \\ & & M & & \end{array}$
$(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$	assoc. law	$\begin{array}{ccc} M \otimes M \otimes M & \longrightarrow & M \otimes M \\ \downarrow & & \downarrow \\ M \otimes M & \longrightarrow & M \end{array}$

inner depends on outer

Formalizing the microcosm principle

What do we mean by "microcosm principle"?

mathematical definition of such nested models?

$$\mathbb{L} \underset{\mathbb{C}}{\overset{\mathbf{1}}{\underset{\Downarrow X}{\rightrightarrows}}} \mathbf{CAT}$$

inner model
as lax natural trans.

algebraic theory
as *Lawvere theory*

outer model
as prod.-pres. functor

Outline

for arbitrary algebraic theory

generic compositionality theorem

2-categorical formulation

$$\mathbb{L} \underset{\mathbb{C}}{\overset{\mathbf{1}}{\underset{\Downarrow X}{\rightrightarrows}}} \mathbf{CAT}$$

microcosm for concurrency (|| and ||)

parallel composition via **sync** nat. trans.

Parallel composition of coalgebras via *sync*

Part 1

Parallel

bifunctor $\mathbf{Coalg}_F \times \mathbf{Coalg}_F \rightarrow \mathbf{Coalg}_F$

usually denoted by (tensor)

Aim

$$\mathbf{beh}\left(\begin{pmatrix} FX \\ c \uparrow \\ X \end{pmatrix} \otimes \begin{pmatrix} FY \\ d \uparrow \\ Y \end{pmatrix}\right) = \mathbf{beh}\left(\begin{pmatrix} FX \\ c \uparrow \\ X \end{pmatrix}\right) \,||\, \mathbf{beh}\left(\begin{pmatrix} FY \\ d \uparrow \\ Y \end{pmatrix}\right)$$

Theorem

: $\mathbf{Coalg}_F \times \mathbf{Coalg}_F \rightarrow \mathbf{Coalg}_F$

***F* with sync**

sync$_{X,Y}$: $FX \quad FY \rightarrow F(X \quad Y)$

lifting

: $\mathbf{C} \times \mathbf{C} \rightarrow \mathbf{C}$

Parallel composition via sync

$\textbf{sync}_{X,Y}$: $FX \quad FY \rightarrow F(X \quad Y)$

$$\begin{pmatrix} FX \\ c\uparrow \\ X \end{pmatrix} \otimes \begin{pmatrix} FY \\ d\uparrow \\ Y \end{pmatrix} = \begin{array}{c} F(X \otimes Y) \\ \uparrow \mathsf{sync}_{X,Y} \\ FX \otimes FY \\ \uparrow c \otimes d \\ X \otimes Y \end{array}$$

on the base category

different sync → different

Examples of sync : $FX \otimes FY \to F(X \otimes Y)$

: $\mathrm{Coalg}_F \times \mathrm{Coalg}_F \to \mathrm{Coalg}_F$

F with sync

lifting

x : Sets x Sets → Sets

- **CSP-style** (Hoare) $a.P \parallel a.Q \xrightarrow{a} P \parallel Q$

$$\begin{array}{ccc} \mathcal{P}_{\mathrm{fin.}}(\Sigma \times X) \times \mathcal{P}_{\mathrm{fin.}}(\Sigma \times Y) & \xrightarrow{\mathrm{sync}_{X,Y}} & \mathcal{P}_{\mathrm{fin.}}(\Sigma \times (X \times Y)) \\ (S,T) & \longmapsto & \{ (a,(x,y)) \mid (a,x) \in S \wedge (a,y) \in T \} \end{array}$$

- **CCS-style** (Milner) $a.P \parallel \overline{a}.Q \xrightarrow{\tau} P \parallel Q$

  Assuming $\Sigma = \{a, a', \dots\} + \{\overline{a}, \overline{a'}, \dots\} + \{\tau\}$

$$\begin{array}{ccc} \mathcal{P}_{\mathrm{fin.}}(\Sigma \times X) \times \mathcal{P}_{\mathrm{fin.}}(\Sigma \times Y) & \xrightarrow{\mathrm{sync}_{X,Y}} & \mathcal{P}_{\mathrm{fin.}}(\Sigma \times (X \times Y)) \\ (S,T) & \longmapsto & \{ (\tau,(x,y)) \mid (a,x) \in S \wedge (\overline{a},y) \in T \} \end{array}$$

C = **Sets**, *F* = $P_{\mathrm{fin}}(\Sigma \times \_)$

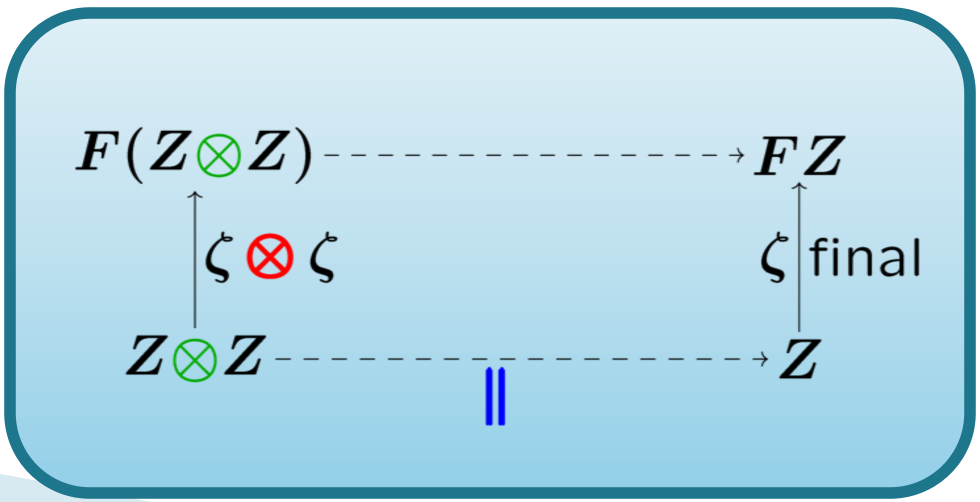
F-coalgebra = LTS

Inner composition

Aim

$$\mathbf{beh}\left(\begin{array}{c} FX \\ c\uparrow \\ X \end{array} \otimes \begin{array}{c} FY \\ d\uparrow \\ Y \end{array}\right) = \mathbf{beh}\left(\begin{array}{c} FX \\ c\uparrow \\ X \end{array}\right) \,\|\, \mathbf{beh}\left(\begin{array}{c} FY \\ d\uparrow \\ Y \end{array}\right)$$

$\|$ "composition of states/*behavior*"
arises by **coinduction**



Compositionality theorem

Theorem

$$\mathrm{beh}\left(\begin{matrix}FX\\ c\uparrow\\ X\end{matrix} \;\otimes\; \begin{matrix}FY\\ d\uparrow\\ Y\end{matrix}\right) \;=\; \mathrm{beh}\left(\begin{matrix}FX\\ c\uparrow\\ X\end{matrix}\right) \,\|\, \mathrm{beh}\left(\begin{matrix}FY\\ d\uparrow\\ Y\end{matrix}\right)$$

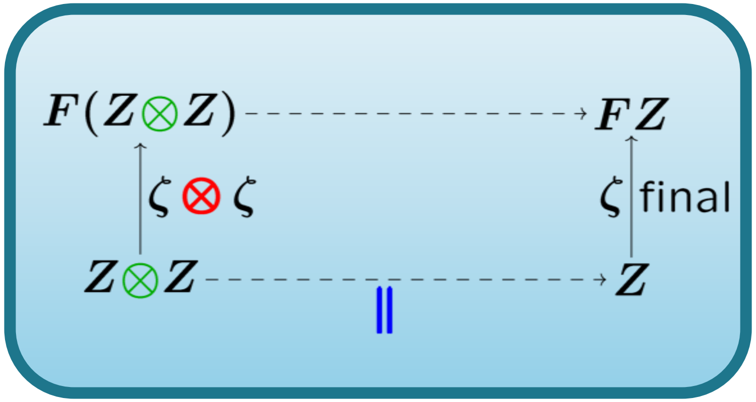
for by and $\|$ by

: $\mathrm{Coalg}_F \times \mathrm{Coalg}_F \rightarrow \mathrm{Coalg}_F$

F with sync

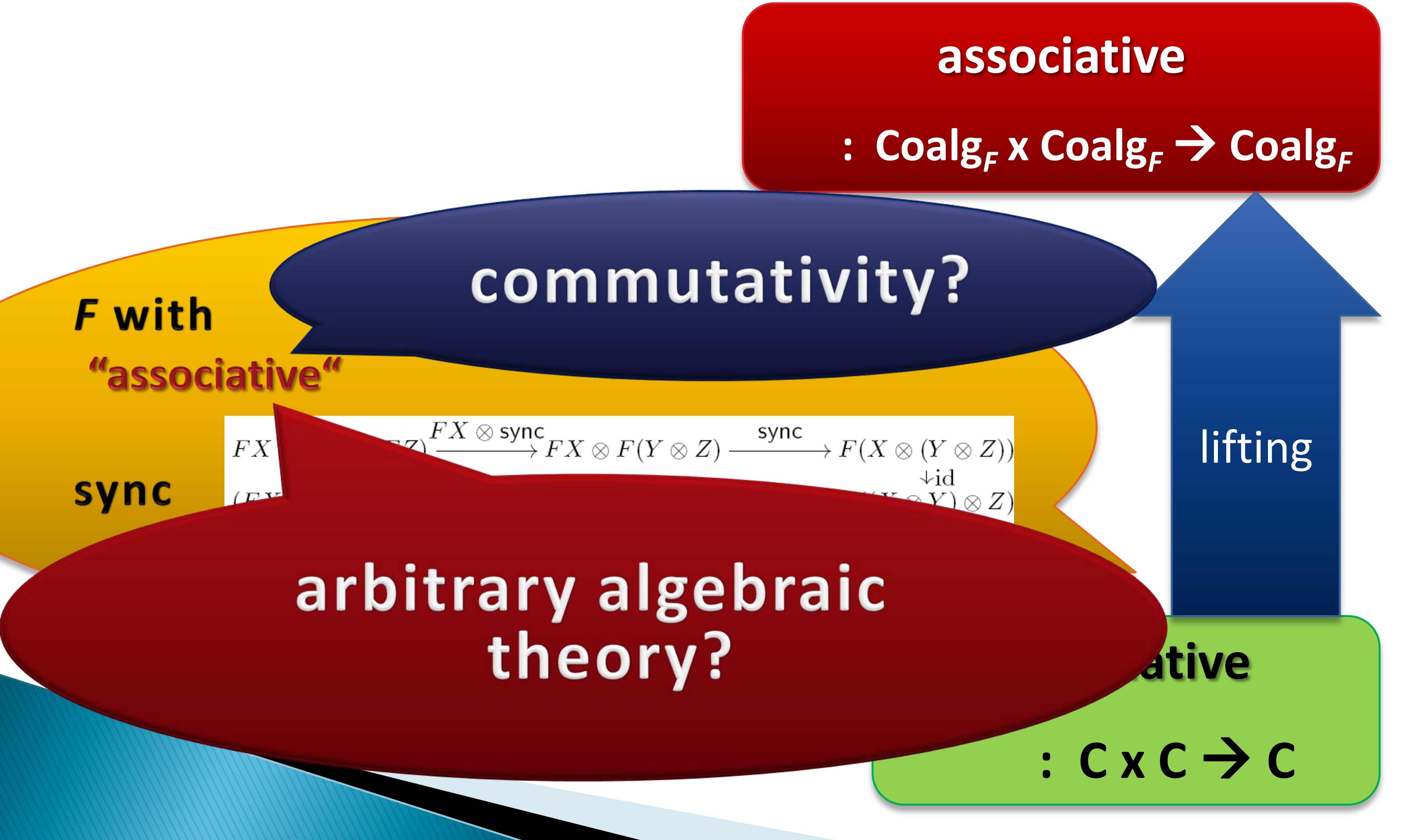
lifting

: C x C → C



Assumptions: , **sync**, final exists

Equational properties



for arbitrary algebraic theory

2-categorical formulation of the microcosm principle

Part 2

Lawvere theory $\mathbb{L}$

a **category** representing an algebraic theory

Definition

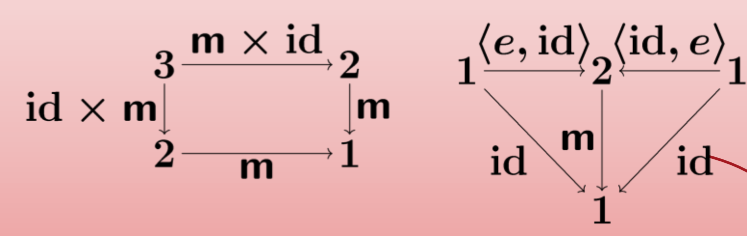
A **Lawvere theory** $\mathbb{L}$ is a small category

- with objects natural numbers
- that has finite products

Lawvere theory

other arrows:

- projections $2 \overset{\pi_1}{\underset{\pi_2}{\rightrightarrows}} 1$
- composed terms $3 \xrightarrow{\mathbf{m}(\mathbf{m}(\pi_1, \pi_2), \pi_3)} 1$

algebraic theory	as category
operations	as **arrows**
m (binary) **e** (nullary)	$2 \xrightarrow{\mathbf{m}} 1$ $0 \xrightarrow{\mathbf{e}} 1$
equations	as **commuting diagrams**
assoc. of **m** unit law	

Models for a Lawvere theory $\mathbb{L}$

Standard: set-theoretic model

a set with $\mathbb{L}$-structure, **$\mathbb{L}$-set**

$$\mathbb{L} \xrightarrow{\ \boldsymbol{X}\ } \mathbf{Sets}$$

(product-preserving)

$$\begin{array}{ccc} \mathbf{2} & & \boldsymbol{X}^{\mathbf{2}} \\ \downarrow \mathbf{m} & \longmapsto & \downarrow [\mathbf{m}] \\ \mathbf{1} & & \boldsymbol{X} \end{array}$$

binary op. on X

what about nested models?

$$X \in \mathbb{C}$$

Outer model: $\mathbb{L}$-category

outer model

- a **category** with $\mathbb{L}$-structure, **$\mathbb{L}$-category**

$$\mathbb{L} \xrightarrow{\quad \mathbb{C} \quad} \mathbf{Cat}$$ (product-preserving)

$$\begin{array}{c} \mathbf{2} \\ \downarrow \mathbf{m} \\ \mathbf{1} \end{array} \longmapsto \begin{array}{c} \mathbb{C}^{\mathbf{2}} \\ \downarrow [\mathbf{m}] = \otimes \\ \mathbb{C} \end{array}$$

Inner model: $\mathbb{L}$-object

Definition

Given an $\mathbb{L}$-category $\mathbb{C}$,

an **$\mathbb{L}$-object** X in it

is a lax natural transformation compatible with products.

inner alg. str.
by
mediating 2-cells

components

$$X_0 : 1 \xrightarrow{\;!\;} 1$$

$$X_1 : 1 \xrightarrow{\;X\;} \mathbb{C}$$

$$X_2 : 1 \xrightarrow{\;(X,X)\;} \mathbb{C}^2$$

$$\vdots$$

X: carrier obj.

$$\frac{X \in \mathbb{C}}{1 \xrightarrow{X} \mathbb{C}}$$

lax naturality

In $\mathbb{L}$

$$\begin{array}{c} \mathbf{2} \\ \downarrow \mathbf{m} \\ \mathbf{1} \end{array}$$

In Cat

$$\begin{array}{ccc} 1 & \xrightarrow{(X,X)} & \mathbb{C}^2 \\ \| & & \downarrow \otimes \\ 1 & \xrightarrow[X]{} & \mathbb{C} \end{array}$$

$$X \otimes X \xrightarrow{X_{\mathbf{m}}} X \quad \text{in } \mathbb{C}$$

lax $\mathbb{L}$-functor

= F with sync

lax *L*-functor?

$\mathbb{L}$ $\mathbb{C}$ $\mathbf{Cat}$ lax natur. trans.

Equations are built in!

Theorem

Coalg$_F$ is an *L*-category

lax *L*-functor *F*

lifting

L-category

lax naturality?

In $\mathbb{L}$

$\mathbf{2} \xrightarrow{\mathbf{m}} \mathbf{1}$

In Cat

$$\begin{array}{ccc} \mathbb{C}^2 & \xrightarrow{(F,F)} & \mathbb{C}^2 \\ \otimes \downarrow & \swarrow & \downarrow \otimes \\ \mathbb{C} & \xrightarrow{F} & \mathbb{C} \end{array}$$

$$FX \otimes FY \xrightarrow{\text{sync}_{X,Y}} F(X \otimes Y) \quad \text{in } \mathbb{C}$$

Theorem

The final object of an L-category is an L-object

Compositionality theorem

Theorem — The behaviour functor *beh* is a strict L-functor

$$\mathrm{Coalg}_F \xrightarrow{\ \mathbf{beh}\ } \mathbb{C}/Z$$

$$\begin{pmatrix} FX \\ c \uparrow \\ X \end{pmatrix} \longmapsto \left(X \xrightarrow{\mathbf{beh}(c)} Z\right)$$

by coinduction

$$\begin{array}{ccc} FX & \dashrightarrow & FZ \\ c \uparrow & & \cong \uparrow \text{final} \\ X & \underset{\mathbf{beh}(c)}{\dashrightarrow} & Z \end{array}$$

In a situation

Coalg_F is an *L*-category

lax *L*-functor *F*

lifting

***L*-category C**

The final object of an L-category is an L-object

Assumptions: **C** is an ***L***-category, ***F*** is lax ***L***-functor, final exists

Related and future work: bialgebras

Bialgebraic structures

[Turi-Plotkin, Bartels, Klin, …]

algebraic structures on coalgebras

In the current work

Equations, not only operations ,are an integral part

The algebraic structures are nested, higher dimensional

Missing

Full GSOS expressivity

Conclusion

Microcosm principle

$X \in \mathbb{C}$

outer interpretation

inner interpretation

algebraic theory

- operations
- equations

2-categorical formulation

$$\mathbb{L} \underset{\mathbb{C}}{\overset{\mathbf{1}}{\underset{\Downarrow X}{\rightrightarrows}}} \mathbf{CAT}$$

Concurrency in coalgebra as motivation and CS example