The Microcosm Principle and Concurrency in Coalgebras

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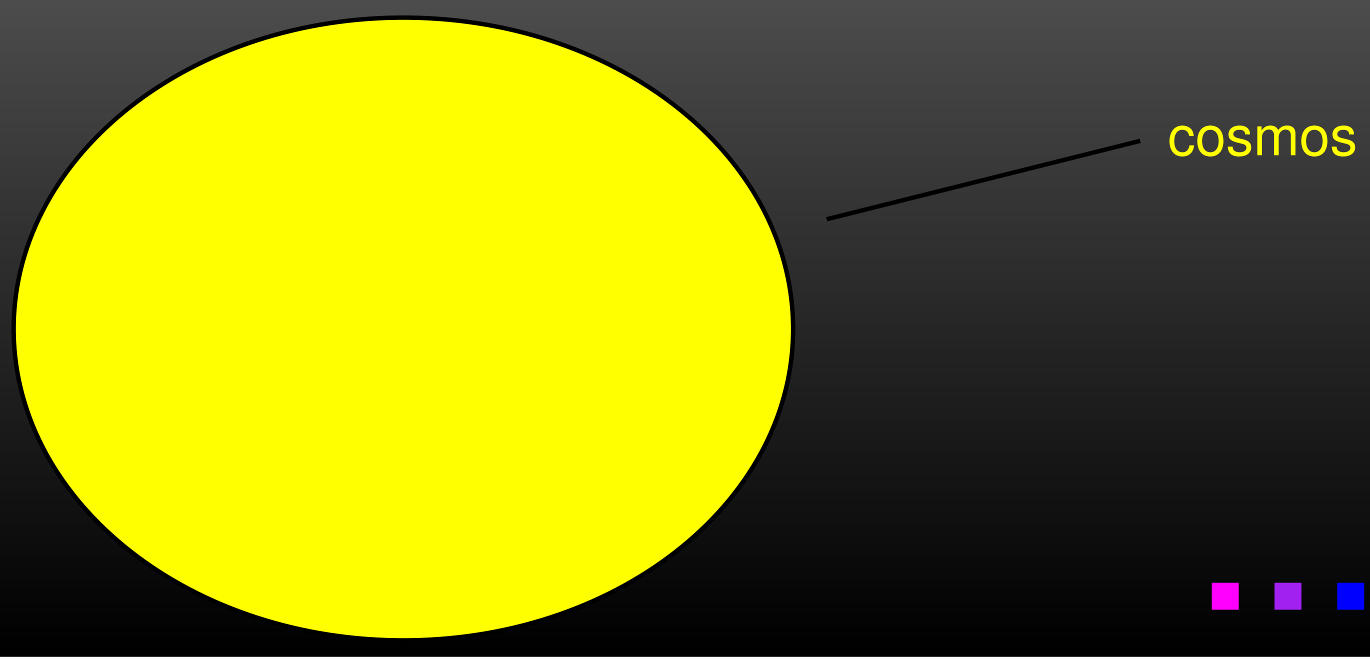
Microcosm principle

(Baez & Dolan)



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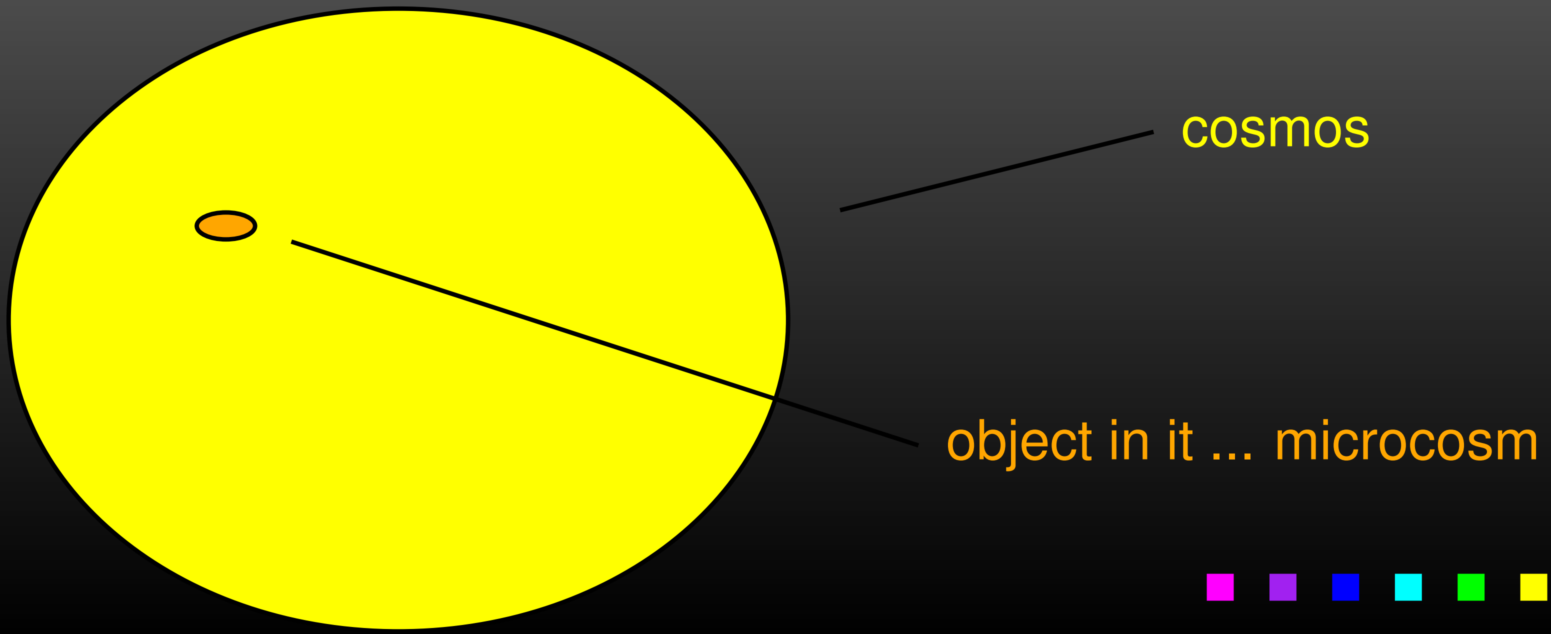
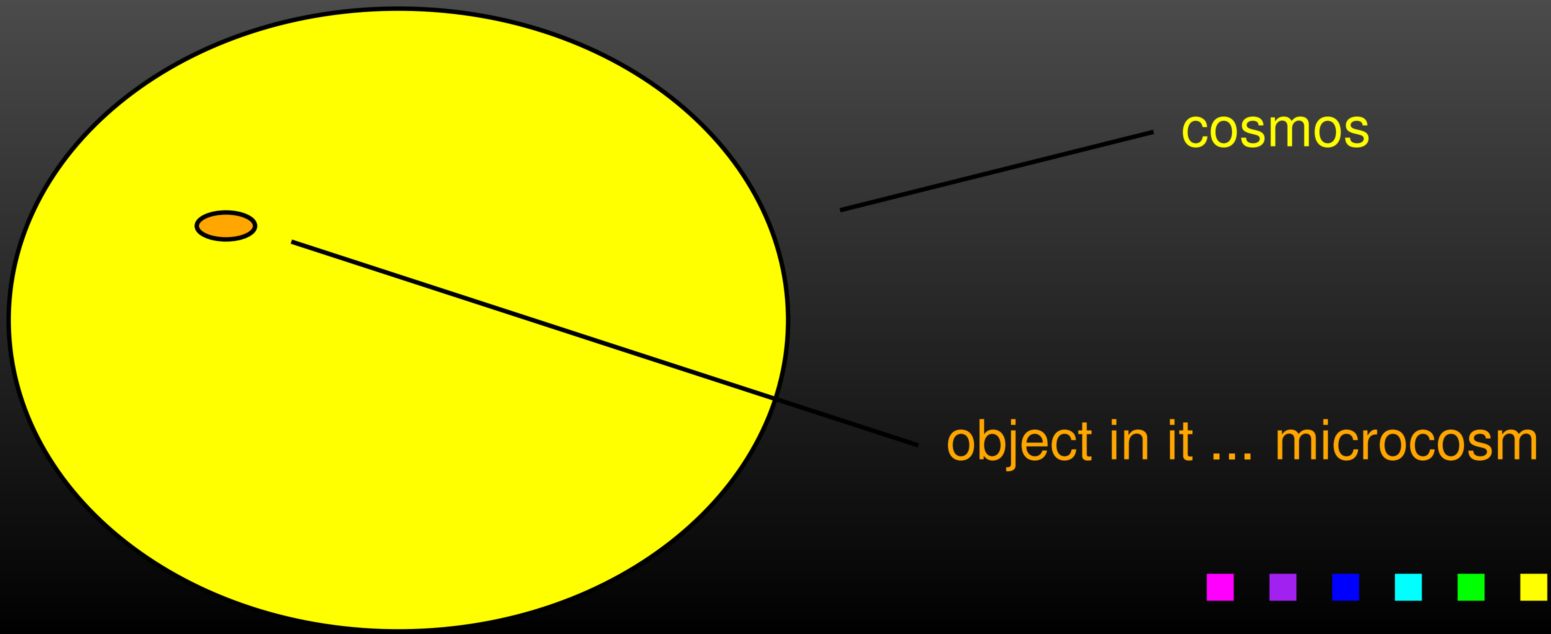


Microcosm principle

(Baez & Dolan)

“A monoid object lives in a monoidal category which is itself a kind of monoid object."



Coalgebras

are an elegant generalization of transition systems with

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$$\langle S, \alpha : S \to \mathcal{F}S \rangle, \text{ for } \mathcal{F} \text{ a functor}$$

- a uniform way for treating transition systems
- general notions and results, e.g. generic notion of bisimulation

Examples of Coalgebras

LTS

$$X \xrightarrow{c} \mathcal{P}(A \times X)$$

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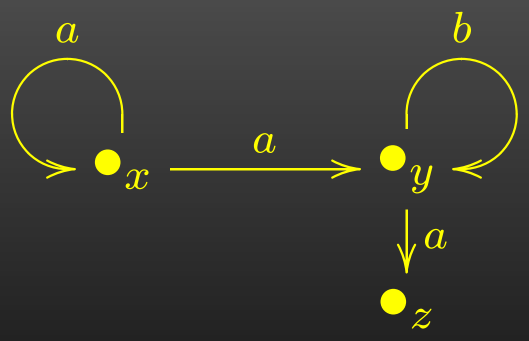

$$c(x) \;=\; \{\langle a, x\rangle, \langle a, y\rangle\}, \;\ldots$$

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(Generative) Probabilistic systems

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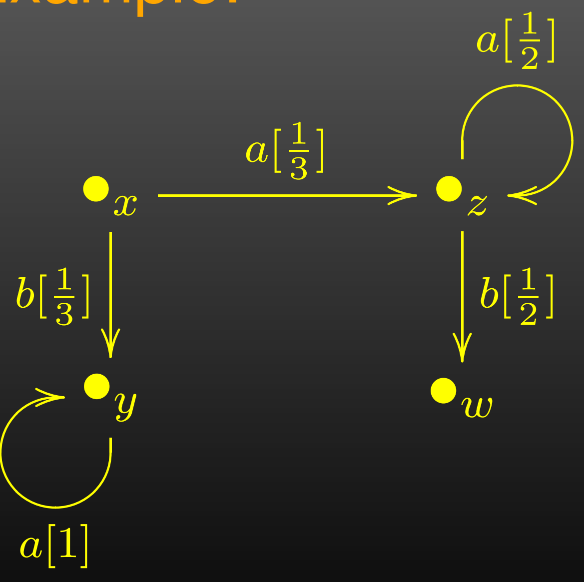

$$c(x) = \{\, \langle b, y\rangle \mapsto \tfrac{1}{3},\ \langle a, z\rangle \mapsto \tfrac{1}{3} \}\,,$$

$$c(y) = \{\, \langle a, y\rangle \mapsto 1 \}\,,\ \ldots$$

Concurrency in coalgebras

Aim: well-behaved concurrency operations on coalgebras

Solution: via nested algebraic structure, microcosm models

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Well-behaved:

* compositional
* associative, commutative,. . .

for instance

LTS, synchronous parallel $|$, with $A(\cdot)$ comm., assoc. partial

$$x \mid y \xrightarrow{a} x' \mid y' \quad \Longleftrightarrow \quad x \xrightarrow{b} x', y \xrightarrow{c} y', a = b \cdot c$$

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? Associativity: $(x \mid y) \mid z \sim x \mid (y \mid z)$

$\otimes$-category

Ingredients:

category $\mathbb{C}$ $\otimes$ α

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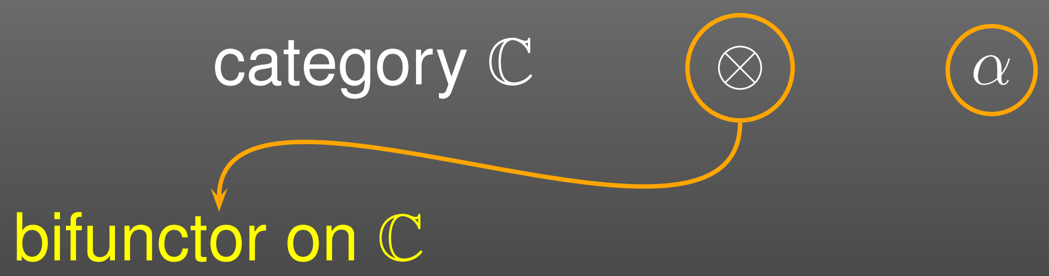

$\otimes$-category

Ingredients:

category $\mathbb{C}$ $\quad \otimes \quad \alpha$

bifunctor on $\mathbb{C}$

nat. iso. $\alpha : X \otimes (Y \otimes Z) \xrightarrow{\cong} (X \otimes Y) \otimes Z$

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Example: $\mathbf{Sets}$ with cartesian product and

$$\alpha(\langle x, \langle y, z\rangle\rangle) = \langle\langle x, y\rangle, z\rangle$$

... is moreover symmetric monoidal

Microcosm phenomenon

“ a $\otimes$-object lives in a $\otimes$-category"

Microcosm phenomenon

A $\otimes$-object or a semigroup in a $\otimes$-category $\mathbb{C}$ is

- an object $S \in \mathbb{C}$
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Microcosm phenomenon

A ⊗-object or a semigroup in a ⊗-category $\mathbb{C}$ is

- an object $S \in \mathbb{C}$
- with a binary operation $m : S \otimes S \to S$
- which is associative

$$\begin{array}{ccccc} S \otimes (S \otimes S) & \xrightarrow{\alpha} & (S \otimes S) \otimes S & \xrightarrow{m \otimes \mathit{id}} & S \otimes S \\ {\scriptstyle \mathit{id} \otimes m}\downarrow & & & & \downarrow{\scriptstyle m} \\ S \otimes S & & \xrightarrow{\qquad\qquad m \qquad\qquad} & & S \end{array}$$

Compatible functors

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$$\begin{array}{ccc} FX \otimes (FY \otimes FZ) & \xrightarrow{\alpha} & (FX \otimes FY) \otimes FZ \\ {\scriptstyle \mathrm{sync} \circ (id \otimes \mathrm{sync})} \downarrow & & \downarrow {\scriptstyle \mathrm{sync} \circ (\mathrm{sync} \otimes id)} \\ F(X \otimes (Y \otimes Z)) & \xrightarrow{F\alpha} & F((X \otimes Y) \otimes Z) \end{array}$$

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Hence - process operations on coalgebras !

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compositionality: $\mathrm{beh}(c \otimes d) \;=\; \| \circ (\mathrm{beh}(c) \otimes \mathrm{beh}(d))$

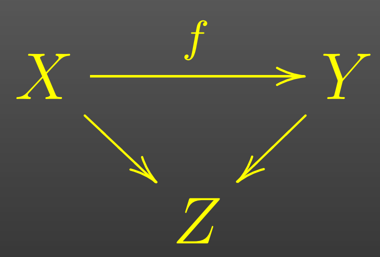
... $\mathrm{beh}(c)$ is obtained by finality ...

Another $\otimes$-category

the slice category $\mathbb{C}/Z$

objects: arrows $X \to Z$ in $\mathbb{C}$

arrows:



commuting triangles

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$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

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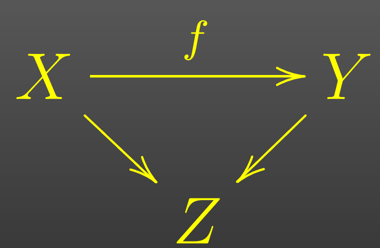

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compositionality is direct here !

All together...

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we have a commuting diagram of $\otimes$-functors between the $\otimes$-categories

$$\begin{array}{ccc} \mathsf{Coalg}_{\mathcal{F}} & \xrightarrow{\;\mathrm{beh}\;} & \mathbb{C}/Z \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle \mathrm{dom}} & \\ & \mathbb{C} & \end{array}$$

all with identity as sync-map

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since the LTS functor is a $\otimes$-functor with

$$\mathrm{sync}_{LTS}(U, V) = \{\langle l, \langle u, v\rangle\rangle \mid \langle a, u\rangle \in U, \langle b, v\rangle \in V, l = a \cdot b\}$$

PTS

the parallel composition

$$x \mid y \stackrel{l[p]}{\longrightarrow} x' \mid y' \qquad \Longleftrightarrow \qquad p = \sum_{q,r:\, l=a\cdot b,\, x \stackrel{a[q]}{\longrightarrow} x',\, y \stackrel{b[r]}{\longrightarrow} y'} q \cdot r$$

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$$\text{sync}_{PTS}(\xi, \psi)(l, \langle x, y \rangle) = \sum_{a,b:\, l=a\cdot b} \xi(a,x) \cdot \psi(b,y)$$

In Kleisli category...

trace semantics is the final coalgebra semantics !

[Hasuo, Jacobs, Sokolova - CMCS'06]

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traces for TF-coalgebras

- monad T - branching type
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now: $\mathcal{K}\ell(T)$ is a $\otimes$-category, if T is a $\otimes$-monad.

hence: compositionality of trace semantics !

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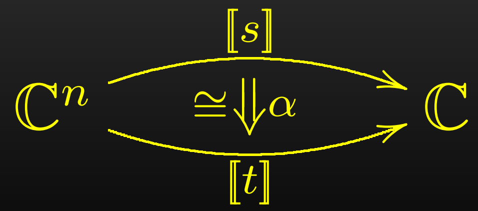
(Σ, E)-cat.: Σ-algebra in $\mathbf{Cat}$ s.t. for $(s = t) \in E$

there is a nat. iso. α:



(Σ, E)-objects

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(Σ, E)-object: $\widehat{\Sigma}$-algebra in $\mathbb{C}$ s.t. for $(s = t) \in E$

$$\begin{array}{ccc} [\![s]\!](A, \ldots, A) & \xrightarrow[\cong]{\alpha} & [\![t]\!](A, \ldots, A) \\ & {\scriptstyle [\![s]\!]} \searrow \quad \swarrow {\scriptstyle [\![t]\!]} & \\ & A & \end{array}$$

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microcosm model of (Σ, E): (Σ, E)-object in a (Σ, E)-category

(Σ, E)-functor

is $F : \mathbb{C} \to \mathbb{D}$ that forms lax coalgebra homomorphism φ:

$$
\begin{array}{ccc}
\coprod_{f \in \Sigma} \mathbb{C}^{|f|} & \xrightarrow{\coprod_{f \in \Sigma} F^{|f|}} & \coprod_{f \in \Sigma} \mathbb{D}^{|f|} \\
\downarrow & \varphi \swarrow & \downarrow \\
\mathbb{C} & \xrightarrow[F]{} & \mathbb{D}
\end{array}
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i.e. a family of natural transformations, for $f \in \Sigma$

$$\varphi^f : [\![f]\!]F^{|f|} \Rightarrow F[\![f]\!]$$

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i.e. a family of natural transformations, for $f \in \Sigma$

$$\varphi^f : [\![f]\!] F^{|f|} \Rightarrow F [\![f]\!]$$

that commutes with the equations for $(s = t) \in E$, α,

$$F\alpha \circ \varphi^s = \varphi^t \circ \alpha$$

Generalized results

Generalized results

given a (Σ, E)-category $\mathbb{C}$ and a (Σ, E)-functor F

the category of F-coalgebras is a (Σ, E)-category

with

$$[\![f]\!](\vec{c_i}) \stackrel{\text{def}}{=} \varphi^f \circ [\![f]\!]_{\mathbb{C}}(\vec{c_i})$$

Generalized results

assume final F-coalgebra $\zeta : Z \xrightarrow{\cong} FZ$ exists

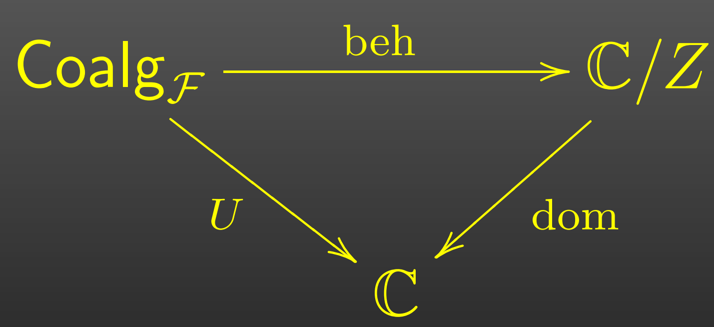
by finality $[\![f]\!]_\zeta : [\![f]\!](Z, \ldots, Z) \to Z$

equations: ζ is a microcosm model in $\mathsf{Coalg}_{\mathcal{F}}$

compositionality: $\mathrm{beh}\,([\![f]\!](\vec{c_i})) = [\![f]\!]_\zeta \circ [\![f]\!]_{\mathbb{C}}(\overrightarrow{\mathrm{beh}(c_i)})$

Generalized results

All together we have a commuting diagram of (Σ, E)-functors between the (Σ, E)-categories



all with identity natural transformations

Generalized results

For compositionality of trace semantics:

If T is a (Σ, E)-monad, then $\mathcal{K}\ell(T)$ is a (Σ, E)-category

Let's generalize further

$(\Sigma, E) \Longrightarrow$ Lawvere theory $\mathbb{L}$

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$\mathbb{L}$-object: a lax natural transformation

$$\mathbb{L} \underset{\mathbb{C}}{\overset{\mathbf{1}}{\rightrightarrows}} \mathbf{Cat} \quad (\Downarrow X)$$

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All results hold - even in greater generality !

Conclusion

- nested structure: microcosm models

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  [on different levels of abstraction]

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