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Naive Set Theory
Relations

Product of multiple sets

Direct product (Kartesisches Produkt)

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

ordered pairs

$$(A \times B) \times C \neq A \times (B \times C)$$

Therefore, we define

$$A \times B \times C = \{(x, y, z) \mid x \in A \text{ and } y \in B \text{ and } z \in C\}$$

if $A_i = A$ for all i ,
then the product is
denoted A^n

sequence of
length n

In general, for sets A_1, A_2, \dots, A_n with $n \geq 1$,

$$A_1 \times A_2 \times \dots \times A_n = \prod_{1 \leq i \leq n} A_i = \{(x_1, x_2, \dots, x_n) \mid x_i \in A_i \text{ for } 1 \leq i \leq n\}$$

Relations

Def. If A and B are sets, then any subset $R \subseteq A \times B$ is a (binary) relation between A and B

similarly, unary relation (subset), n-ary relation...

Def. R is a relation on A if $R \subseteq A \times A$

some relations are special

Special relations

A relation $R \subseteq A \times A$ is:

reflexive	iff	for all $a \in A$, $(a,a) \in R$
symmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \in R$
transitive	iff	for all $a,b,c \in A$, if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$
irreflexive	iff	for all $a \in A$, $(a,a) \notin R$
antisymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$ and $(b,a) \in R$ then $a = b$
asymmetric	iff	for all $a,b \in A$, if $(a,b) \in R$, then $(b,a) \notin R$
total	iff	for all $a,b \in A$, $(a,b) \in R$ or $(b,a) \in R$

(infix) notation aRb for $(a,b) \in R$

Special relations

A relation R on A , i.e., $R \subseteq A \times A$ is:

- | | | |
|-------------------------|-----|---|
| equivalence | iff | R is reflexive, symmetric, and transitive |
| partial order | iff | R is reflexive, antisymmetric, and transitive |
| strict order | iff | R is irreflexive and transitive |
| preorder | iff | R is reflexive and transitive |
| total (linear)
order | iff | R is a total partial order |

Obvious properties

1. Every partial order is a preorder.
2. Every total order is a partial order.
3. Every total order is a preorder.
4. If $R \subseteq A \times A$ is a relation such that there are $a, b \in A$ with
 $a \neq b, (a,b) \in R$ and $(b,a) \in R$,
then R is not a partial order, nor a total order, nor a strict order.

Operations on relations

Let $R \subseteq A \times \underline{B}$ and $S \subseteq \underline{B} \times C$ be two relations. Their composition is the relation

$$R \circ S = \{(a,c) \in A \times C \mid \text{there is } b \in B \text{ s.t. } (a,b) \in R \text{ and } (b,c) \in S\}$$

relational composition is associative $(R \circ S) \circ T = R \circ (S \circ T)$

so again we write
 $R^n = \underbrace{R \circ R \circ \dots \circ R}_{n \text{ times}}$

Let $R \subseteq A \times B$ be a relation. The inverse relation of R is the relation

$$R^{-1} = \{(b,a) \in B \times A \mid (a,b) \in R\}$$

Characterizations

Lemma: Let R be a relation over the set A . Then

1. R is reflexive iff $\Delta_A \subseteq R$
2. R is symmetric iff $R \subseteq R^{-1}$
3. R is transitive iff $R^2 \subseteq R$

Important equivalence on \mathbb{Z}

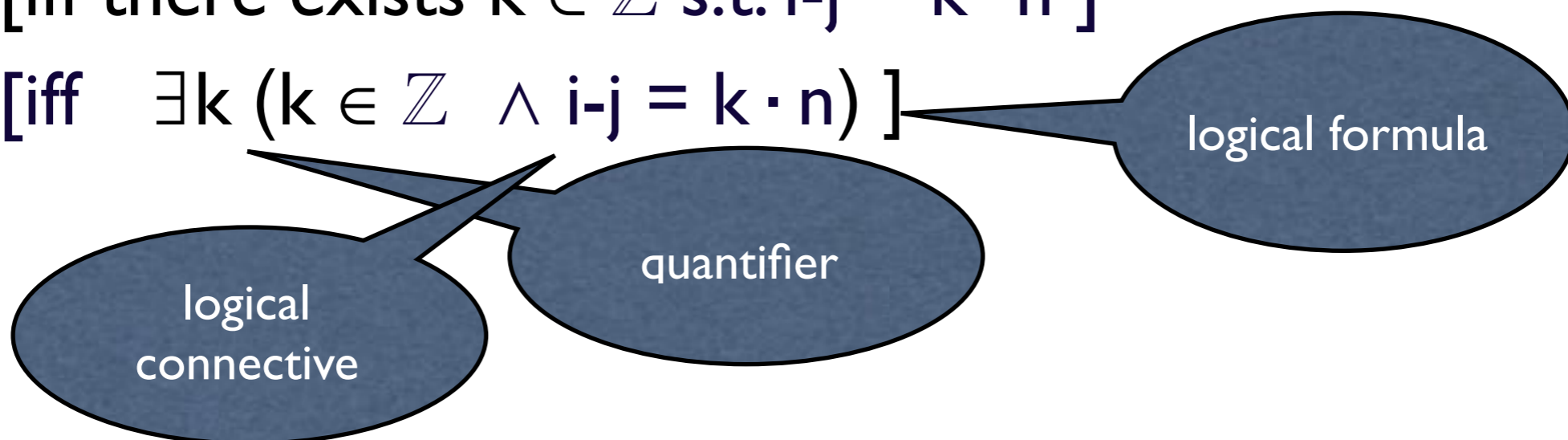
Def. For a natural number n , the relation \equiv_n is defined as

$$i \equiv_n j \quad \text{iff } n \mid i - j$$

[iff $i-j$ is a multiple of n]

[iff there exists $k \in \mathbb{Z}$ s.t. $i-j = k \cdot n$]

[iff $\exists k (k \in \mathbb{Z} \wedge i-j = k \cdot n)$]

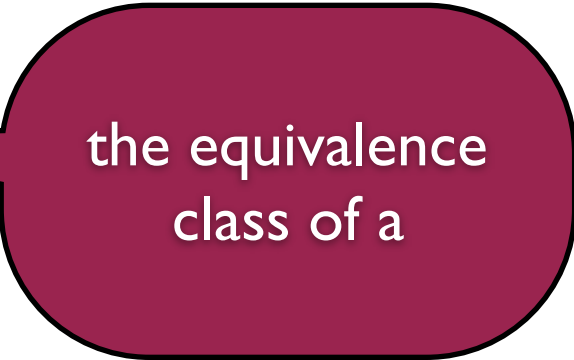


Lemma: The relation \equiv_n is an equivalence for every n .

Equivalences classes

Def. Let R be an equivalence over A and $a \in A$. Then

$$[a]_R = \{ b \in A \mid (a, b) \in R \}$$



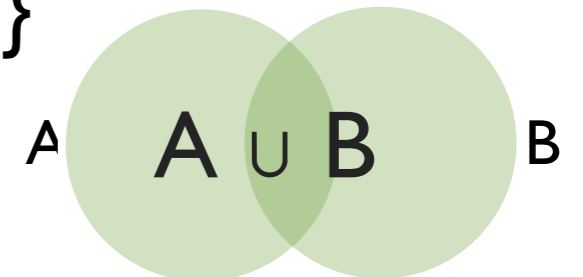
the equivalence
class of a

Lemma: Let R be an equivalence over the set A . Then
for all $a, b \in A$, $[a]_R = [b]_R$ or $[a]_R \cap [b]_R = \emptyset$

Task: Describe the equivalence classes of \equiv_n
How many classes are there?

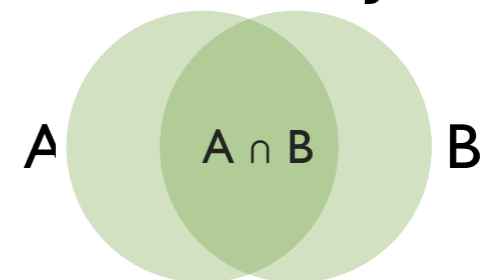
Unions and intersections of multiple sets

Union (**Vereinigung**) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



Intersection (**Durchschnitt**) $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

A and B are **disjoint** if $A \cap B = \emptyset$



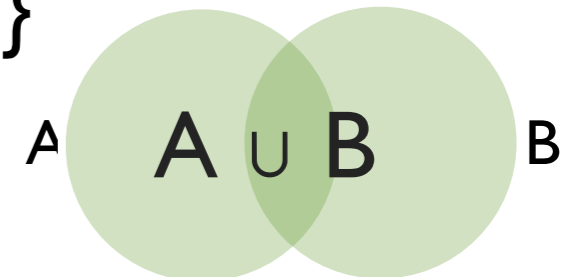
In general, for sets A_1, A_2, \dots, A_n with $n \geq 1$,

$$A_1 \cup A_2 \cup \dots \cup A_n = \bigcup_{1 \leq i \leq n} A_i = \{x \mid x \in A_i \text{ for some } i \in \{1, \dots, n\}\}$$

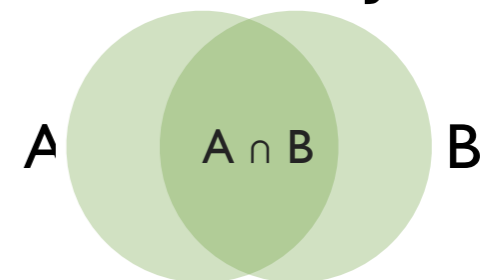
$$A_1 \cap A_2 \cap \dots \cap A_n = \bigcap_{1 \leq i \leq n} A_i = \{x \mid x \in A_i \text{ for all } i \in \{1, \dots, n\}\}$$

Unions and intersections of multiple sets

Union (**Vereinigung**) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$



Intersection (**Durchschnitt**) $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$



A and B are **disjoint** if $A \cap B = \emptyset$

In general, for a **family of sets** $(A_i \mid i \in I)$

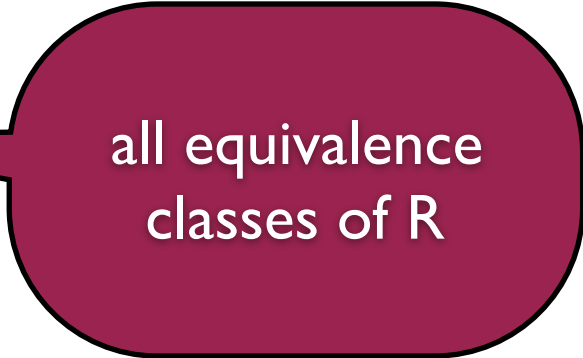
$$\bigcup_{i \in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$

$$\bigcap_{i \in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

Back to equivalence classes

Example: Let R be an equivalence over A and $a \in A$. Then

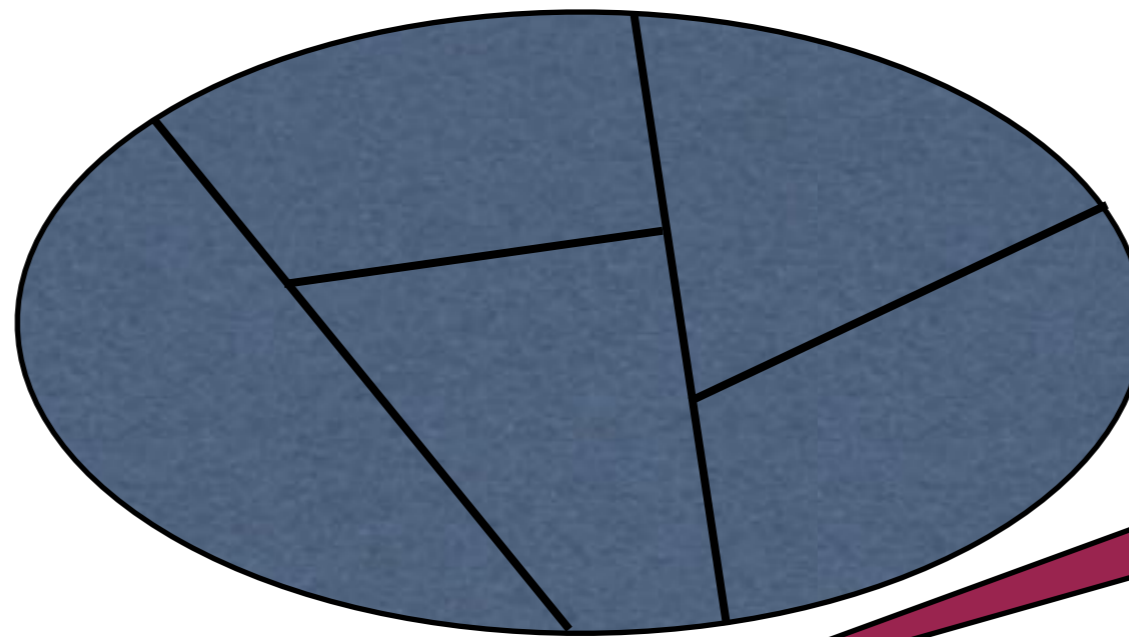
$([a]_R, a \in A)$ is a family of sets.



all equivalence classes of R

Lemma E2: $A = \bigcup_{a \in A} [a]_R$. The union is disjoint.

Partitions



hence, a collection
of
subsets of X

Def. Let X be a set. A subset P of the powerset $\mathcal{P}(X)$ is a partition (**Klasseneinteilung**) of X if it satisfies:

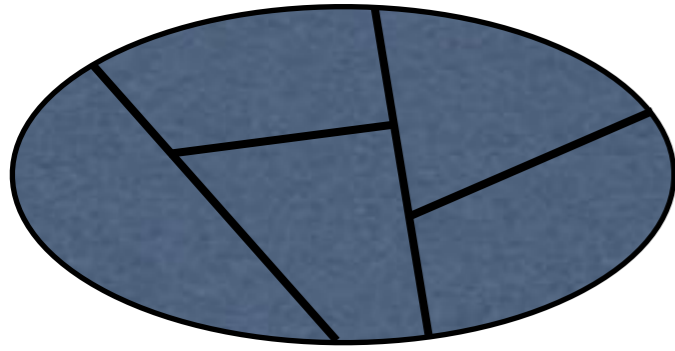
(1) For all $A \in P$, $A \neq \emptyset$

(2) For all $A, B \in P$, if $A \neq B$

then $A \cap B = \emptyset$

(3) $\bigcup_{A \in P} A = X$

that are non-empty,
pairwise disjoint,
and their union equals X



Partitions = Equivalences

Theorem PE: Let X be a set.

(1) If R is an equivalence on X , then the set

$$P(R) = \{ [x]_R \mid x \in X \}$$

is a partition of X .

(2) If P is a partition of X , then the relation

$$R(P) = \{ (x, y) \in X \times X \mid \text{there is } A \in P \text{ such that } x, y \in A \}$$

is an equivalence relation.

Moreover, the assignments $R \mapsto P(R)$ and $P \mapsto R(P)$ are inverse to each other, i.e., $R(P(R)) = R$ and $P(R(P)) = P$.

Transitive closure

Let R be a relation on a set X . The transitive closure (**transitive Hülle**) of R , notation R^+ , is the relation

$$R^+ = \bigcup_{n \in \mathbb{N}, n \neq 0} R^n$$


$$R^{n+1} = R^n \circ R$$

The reflexive and transitive closure (**reflexive und transitive Hülle**) of R , notation R^* , is the relation

$$R^* = \bigcup_{n \in \mathbb{N}} R^n$$


$$R^0 = \Delta_R$$

Proposition TC: Let R be a relation on X . The transitive closure of R is the smallest transitive relation that contains R . The reflexive and transitive closure of R is the smallest reflexive and transitive relation that contains R .