Predicate logic

Limitations of propositional logic

Propositional logic only allows us to reason about completed statements about things, not about the things themselves.

Example

Some chicken cannot fly
All chicken are birds

Some birds cannot fly

this reasoning can not be expressed in propositional logic

Example

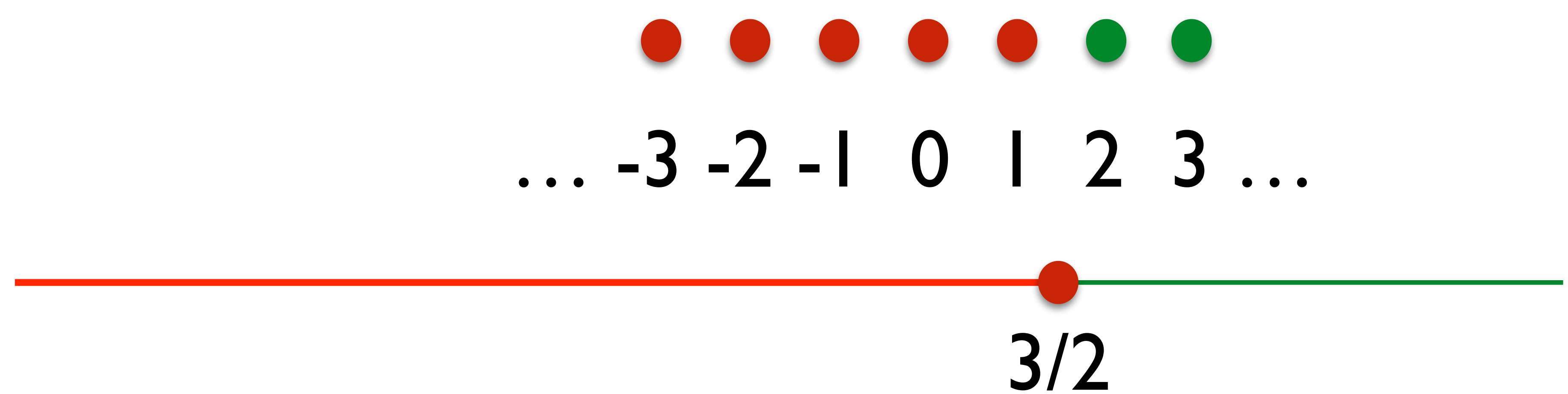
Every player except the winner looses a match

Unary predicate (example)

Consider the statement $2m>3$.

a unary relation

Whether this statement is true or false depends on the value of m (and on the domain of values).



on $\mathbb{Z}$

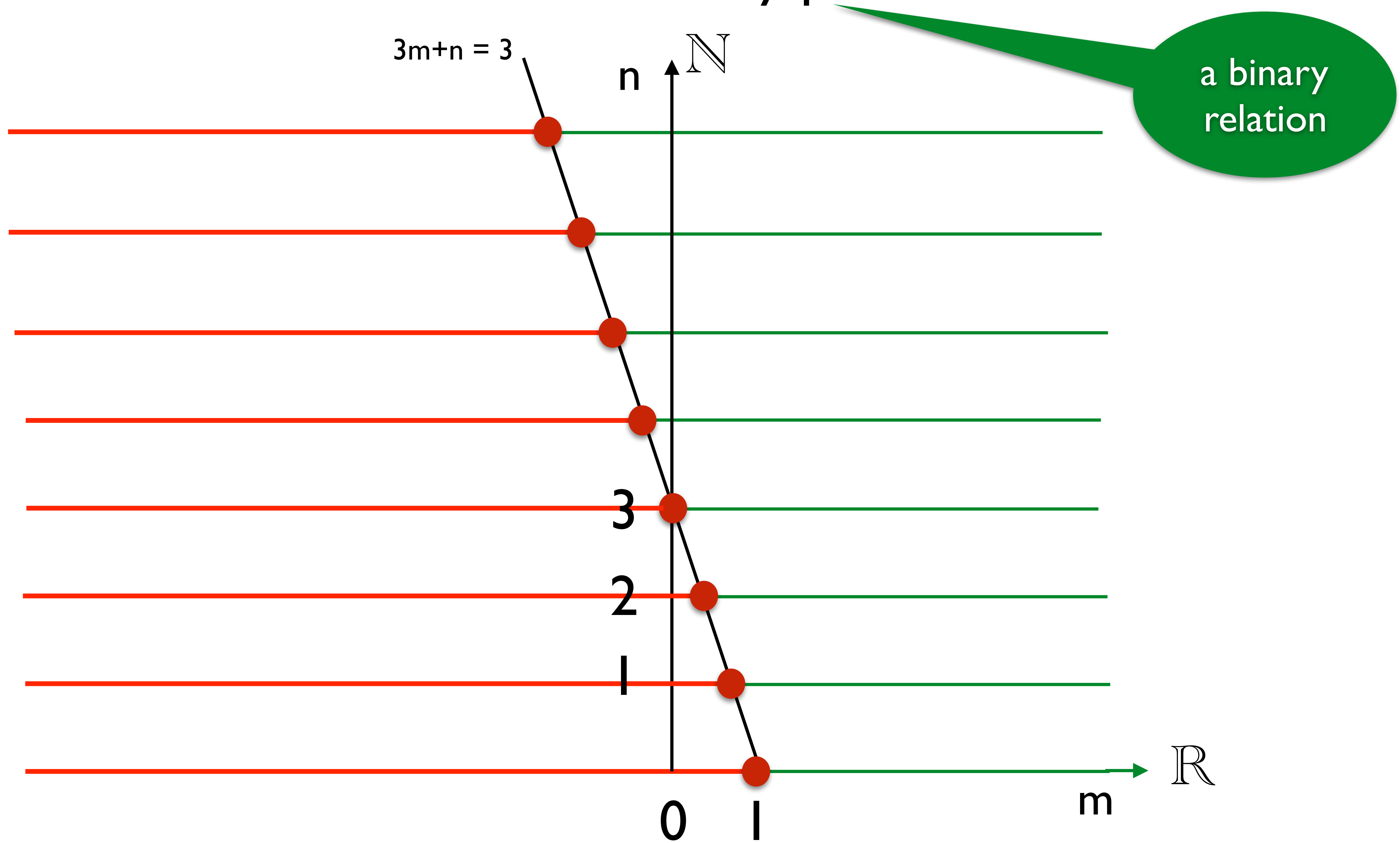
on $\mathbb{R}$

Note: $2m > 3 \overset{\text{val}}{=} m > 3/2$ on $\mathbb{Z}$ and $\mathbb{R}$

$2m > 3 \overset{\text{val}}{=} m \geq 2$ on $\mathbb{Z}$ but not on $\mathbb{R}$

Binary predicate (example)

The statement 3m+n > 3 is a binary predicate on $\mathbb{R}$ x $\mathbb{N}$.



Predicates

In general, an n-ary predicate is an n-ary relation.

If it is on a domain D, then it's a relation $P(x_1, .., x_n) \subseteq D^n$ or equivalently a function $P: D^n \rightarrow \{0,1\}$.

true for certain values of the variables

$2m>3$

We can turn a predicate, into a proposition in three ways:

1. By assigning values to the variables.
2. By universal quantification.
3. By existential quantification.

for $m=2$

$2 \cdot 2 > 3$

is a true proposition

Universal quantification

The unary predicate 2m > 3 on $\mathbb{Z}$ can be turned into a proposition by universal quantification:

false, e.g. for m = 1

For all m in $\mathbb{Z}$, 2m > 3

universal quantifier

standard (!) notation:

$\forall x\ (P(x) \Rightarrow Q(x))$

$\forall x.\ P(x) \Rightarrow Q(x)$

Notation: $\forall_m\ [m \in \mathbb{Z} : 2m > 3]$

domain (predicate)

predicate

In general: $\forall_x\ [P(x) : Q(x)]$ for "all x satisfying P satisfy Q"

Existential quantification

The unary predicate 2m > 3 on $\mathbb{Z}$ can also be turned into a proposition by existential quantification:

true, e.g. m =2

There exists m in $\mathbb{Z}$, 2m > 3

existential quantifier

standard (!) notation:

$\exists x\ (P(x) \wedge Q(x))$

$\exists x.\ P(x) \wedge Q(x)$

Notation: $\exists_m\ [m \in \mathbb{Z} : 2m > 3]$

domain (predicate)

predicate

In general: $\exists_x\ [P(x) : Q(x)]$ for

"there exists x satisfying P that satisfies Q"

Quantification

The binary predicate $3m+n > 3$ on $\mathbb{R} \times \mathbb{N}$ can also be turned into a proposition by quantification:

in 8 possible ways

One way is: $\exists_m\, [m \in \mathbb{R} : \forall_n\, [n \in \mathbb{N} : 3m + n > 3]]$

binary predicate

unary predicate

proposition, nullary predicate

standard (!) notation:

$\exists m\, (m \in \mathbb{R} \land \forall n\, (n \in \mathbb{N} \Rightarrow 3m+n>3))$

Notation

also for $\exists$

We write $\forall_x$ [P] for $\forall_x$ [T : P]

We also write $\exists_{m,} \forall_n$ [(m,n)$\in \mathbb{R}$ x $\mathbb{N}$: 3m + n >3]

for $\exists_m$ [m$\in \mathbb{R}$: $\forall_n$ [n$\in \mathbb{N}$: 3m + n >3]]

And even $\exists_{m, n}$ [(m,n)$\in \mathbb{R}$ x $\mathbb{N}$: 3m + n >3]

for $\exists_m$ [m$\in \mathbb{R}$: $\exists_n$ [n$\in \mathbb{N}$: 3m + n >3]]

but only for the same quantifier!

Quantification - task

Thanks to Bas Luttik

Let P be the set of all tennis players.
Let $w \in P$ be the winner.

For $p, q \in P$, write $p \neq q$ for "p and q are different players" .

Let M be the set of all matches.
For $p \in P$ and $m \in M$, write $L(p,m)$ for
"player p loses match m".

Write the following sentence as a formula with predicates and quantifiers:

Every player except the winner loses a match.

Equivalences with quantifiers

Renaming bound variables

Bound variables

$$\forall_x[P{:}Q] \overset{val}{=} \forall_y[P[y/x]{:}Q[y/x]]$$

$$\exists_x[P{:}Q] \overset{val}{=} \exists_y[P[y/x]{:}Q[y/x]]$$

if y does not occur in
P or Q (not even in ∀y, ∃y)