# Formale Systeme Proseminar

## Tasks for Week 9

**Task 1** Write the following sentences as formulas with quantifiers. $D$ is a subset of $\mathbb{R}$.

(a) All elements of $D$ are not equal to 0.

(b) All elements of $D$ are greater than 10 and smaller than 25.

(c) All elements of $D$ are greater than 10 or all elements of $D$ are smaller than 25.

(d) Every pair of different elements of $D$ differ by at least 1.

**Task 2** Write the following sentences as formulas with quantifiers.

(a) For all natural numbers, there is a natural number which is greater than it by 1.

(b) There is no natural number which is greater than all natural numbers.

(c) The sum of two natural numbers is greater than or equal to each of these two numbers.

(d) There are two natural numbers the sum of whose squares is 58.

**Task 3** Write each of the following propositions as a formula with quantifiers. You may use that $\mathbb{P}$ denotes the set of all prime numbers.

(a) All prime numbers are even, except the number 3.

(b) Every sum of three prime numbers is also a prime number.

(c) There is a prime number which is 1 plus a multiple of 7.

Is the proposition true? If yes, give an explanation; if not, give a counter example.

**Task 4** Check which of the following propositions are equivalent, independently of $D$, where $D$ is an arbitrary subset of $\mathbb{R}$.

(a) $\exists_x[x \in D : \forall_y[y \in D : y \leq x]]$

(b) $\exists_l[l \in D : \forall_k[k \in D : l \leq k]]$

(c) $\exists_k[k \in D : \forall_m[m \in D : \neg(k < m)]]$

(c) $\forall_y[y \in D : \exists_x[x \in D : y \leq x]]$

**Task 5** Show with a counterexample that:

(a) $\forall_k[P : Q] \overset{val}{\neq} \forall_k[Q : P]$

(b) $\exists_k[P : Q] \wedge \exists_k[P : R] \overset{val}{\neq} \exists_k[P : Q \wedge R]$.

**Task 6** Prove with a calculation that the following formula is a tautology.

$$\exists_x[P \wedge \neg R] \Rightarrow \neg\forall_x[P \vee Q : R]$$

**Task 7** Prove with a calculation that the following formula is a tautology.

$$\neg\forall_x[P : P] \vee \exists_x[P \wedge Q : \neg P] \Leftrightarrow \neg\forall_x[P : P]$$

**Task 8** Is the following statement true? If yes, prove it with a calculation; if not, give a counter example.

$$\neg\forall_x[P : Q \wedge R] \overset{val}{=} \neg(\forall_x[P : \neg Q] \wedge \forall_x[P : \neg R])$$

**Task 9** Show that the following formulas are tautologies.

(a) $\neg\forall_x[P \wedge Q : R] \Leftrightarrow \exists_x[P : Q \wedge \neg R]$

(b) $\neg\exists_x[\neg P \vee \neg Q : R] \Leftrightarrow \forall_x[R : P] \wedge \forall_x[R : Q]$