Formale Systeme Proseminar

Tasks for Week 13

As for some of you cardinals may be confusing and difficult, to each task I added "(Note: ...)" which recalls some of the definitions and clarifies what you should really do. I hope this is of help. The actual tasks are as given without the "(Note: ...)" part.

Don't forget to look on page 2 (there are two more tasks :-)).

Task 1 Prove by induction that

$$\forall n[n \in \mathbb{N} \wedge n > 1 : 1 + 3 + \ldots + (2n-1) = n^2].$$

Task 2 The sequence $(a_i \mid i \in \mathbb{N})$ is inductively defined by

$a_0 = 0$

$a_{i+1} = a_i + 3$

Prove (by induction) that $\forall n[n \in \mathbb{N} : 3|a_n]$. Try to find a closed formula for a_n and prove by induction that it is really true.

Task 3 The sequence $(a_i \mid i \in \mathbb{N})$ is inductively defined by

$a_0 = 1$

$a_{i+1} = \frac{1}{i+1} \sum_{k=0}^{i} a_k$

Prove (by induction) that $\forall n[n \in \mathbb{N} : a_n = 1]$.

Task 4 Consider the sets $A = \{a, b\}$ and $B = \{1, 2, 3\}$. Convince yourself on this very example that $|A^B| = |A|^{|B|}$, i.e., that $|A^B| = 8$.

Task 5 Prove that

(a) $A \subseteq B \Rightarrow |A| \leq |B|$.
(Note: You need to construct an injection from A to B.)

(b) $|A| \leq |B| \wedge |B| \leq |C| \Rightarrow |A| \leq |C|$.
(Note: Given two injections $i_1 \colon A \to B$ and $i_2 \colon B \to C$ you need to find an injection $i \colon A \to C$)

Task 6 Prove that any subset of a finite set is finite.
(Note: you need to show that if A is a finite set, i.e., there is a bijection $f \colon A \to \mathbb{N}_k$ for some $k \in \mathbb{N}$ and $B \subseteq A$, then there is a bijection $g \colon B \to \mathbb{N}_m$ for some $m \in \mathbb{N}$.)

Task 7 Let $A_{m,n} = \{k \in \mathbb{N} \mid n \leq k \leq m\}$. Prove that $A_{m,n}$ is a finite set, by explicitly defining a bijection (to one of the sets $\mathbb{N}_l$ for some $l \in \mathbb{N}$).

Task 8 Prove by induction that if A is a finite set, i.e., $|A| = k$ for some $k \in \mathbb{N}$ then
$$|\mathcal{P}(A)| = 2^k.$$
(Note: We proved this property in general in class (for arbitrary cardinals), but for finite cardinals (natural numbers), we can prove it concretely using induction and here 2^k is a natural number, the number of elements in $|\mathcal{P}(A)|$).

Task 9 Prove that for arbitrary cardinals α and β we have $\alpha \cdot \beta = \beta \cdot \alpha$. (Note: $\alpha = |A|$ for some set A, $\beta = |B|$ for some set B. Also, $\alpha \cdot \beta = |A \times B|$ and $\beta \cdot \alpha = |B \times A|$. So your task here is to give a bijection from $A \times B$ to $B \times A$.)