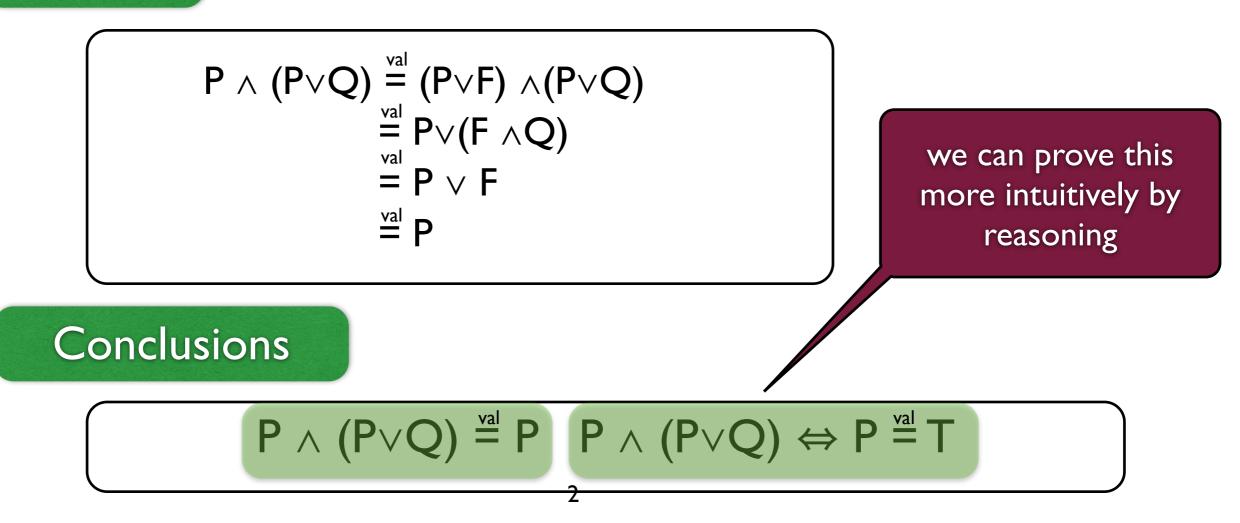
Derivations / Reasoning

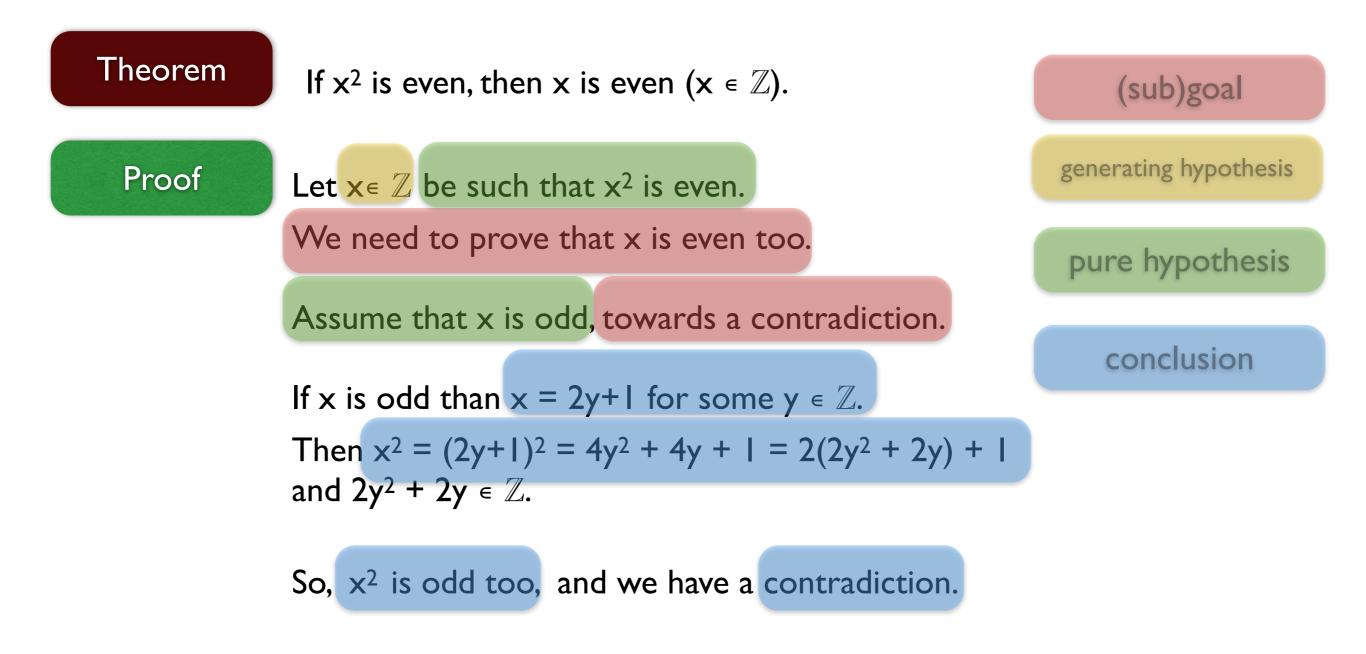
Limitations of proofs by calculation

Proofs by calculation are formal and well-structured, but often undirected and not particularly intuitive.

Example

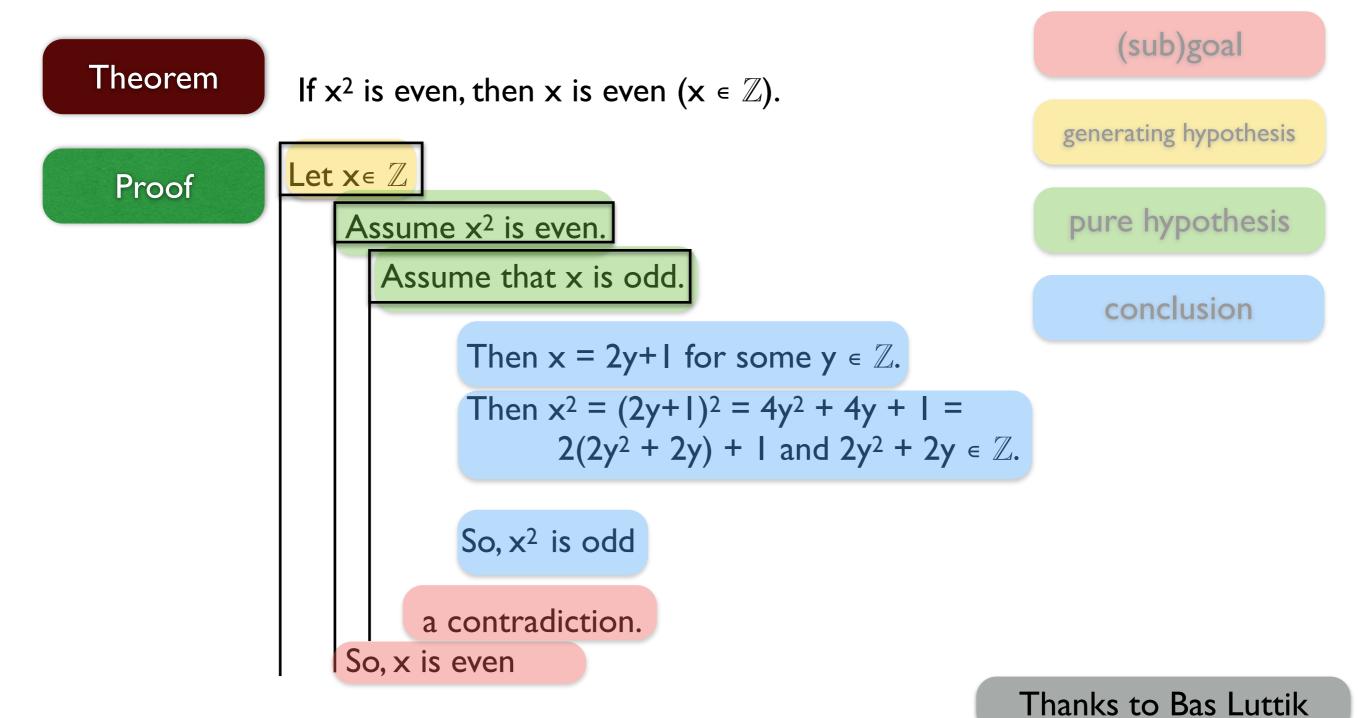


An example of a mathematical proof



Thanks to Bas Luttik

Exposing logical structure



Single inference rule

Q is a correct conclusion from n premises $P_1, ..., P_n$ iff $(P_1 \land P_2 \land ... \land P_n) \stackrel{val}{\vDash} Q$

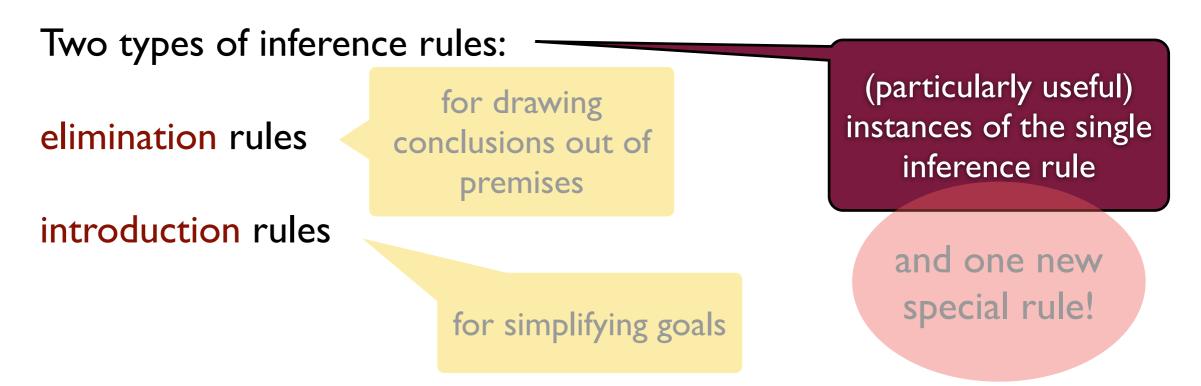
If n=0, then
$$P_1 \wedge P_2 \wedge ... \wedge P_n \stackrel{val}{=} T$$

Note that $T \vDash Q$ means that $Q \stackrel{val}{=} T$ — Q holds unconditionally

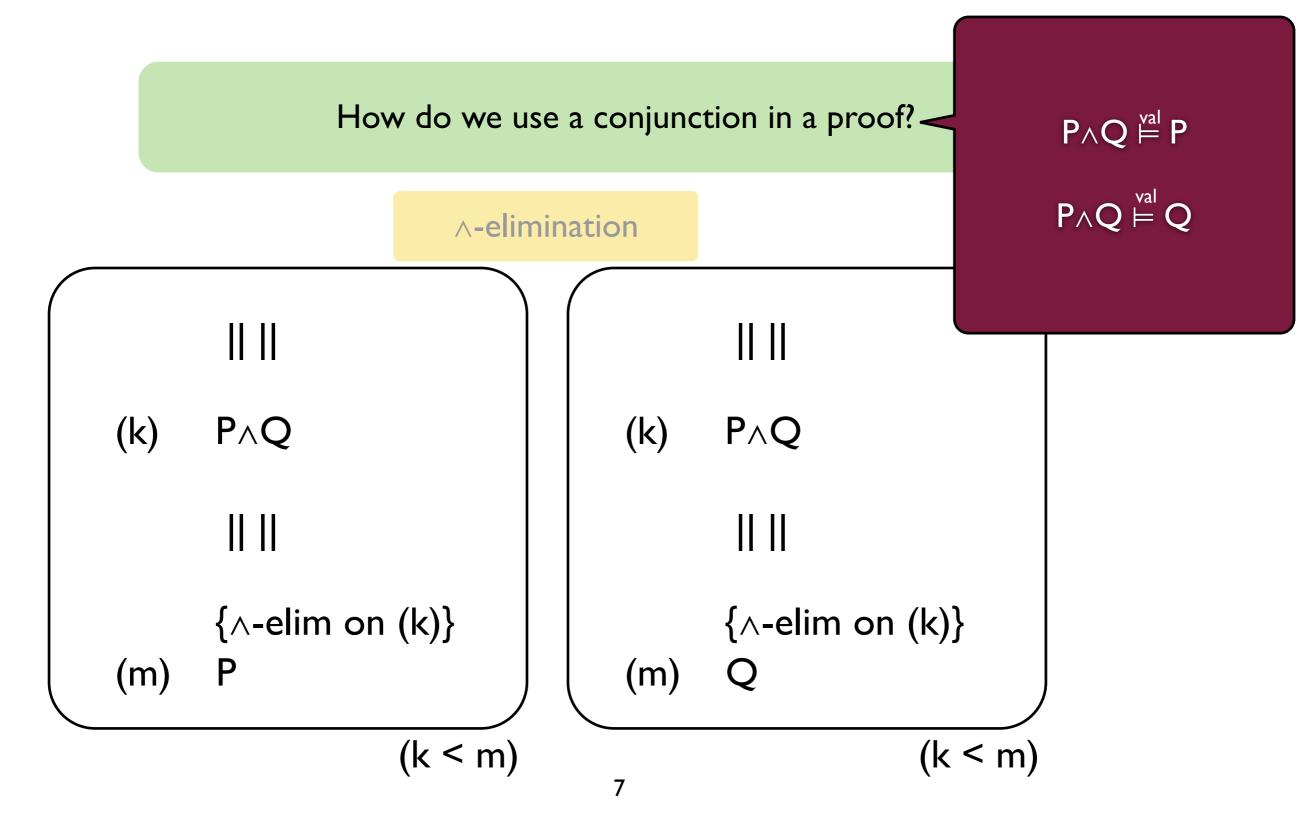
Derivation

 $\begin{array}{l} Q \text{ is a correct conclusion from n premises } P_1, \dots, P_n \\ & \quad \text{iff} \\ \left(P_1 \wedge P_2 \ \wedge \dots \wedge P_n\right) \stackrel{\text{val}}{\vDash} Q \end{array}$

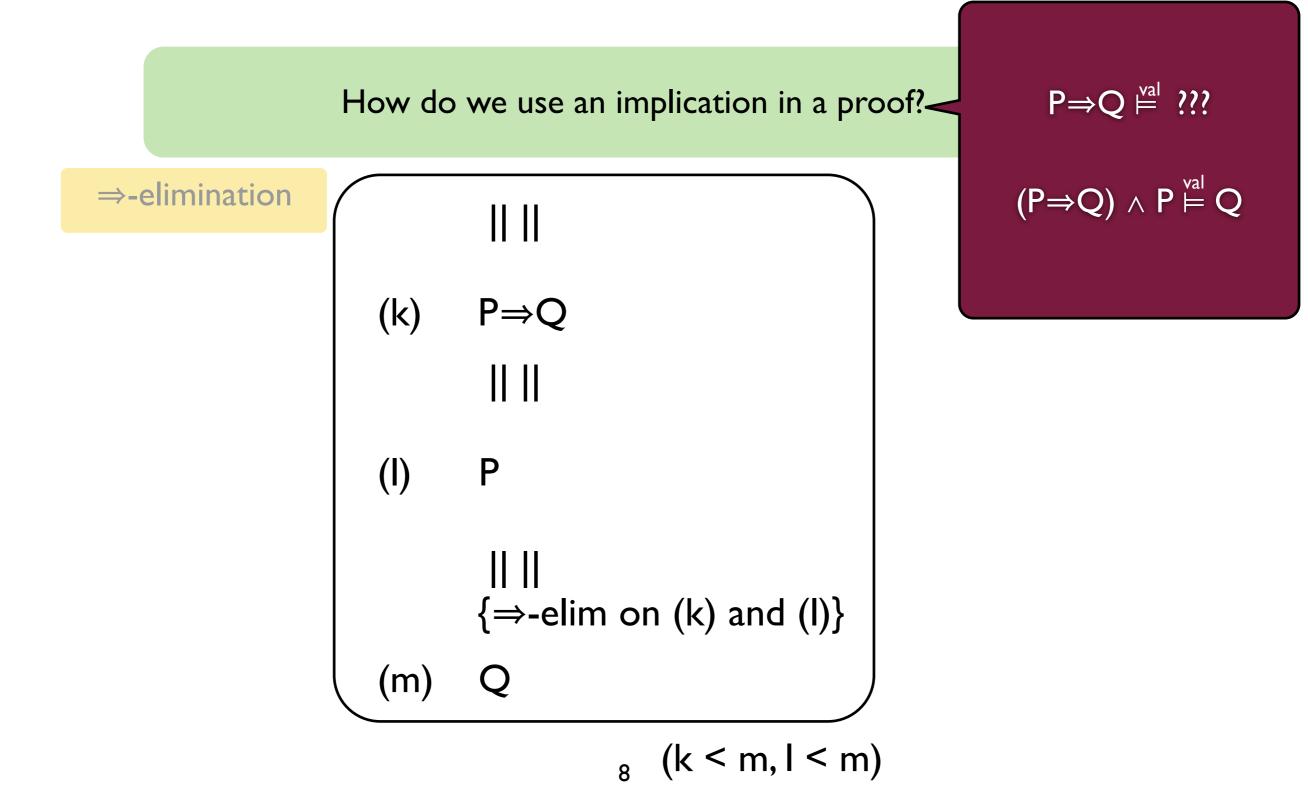
a formal system based on the single inference rule for proofs that closely follow our intuitive reasoning



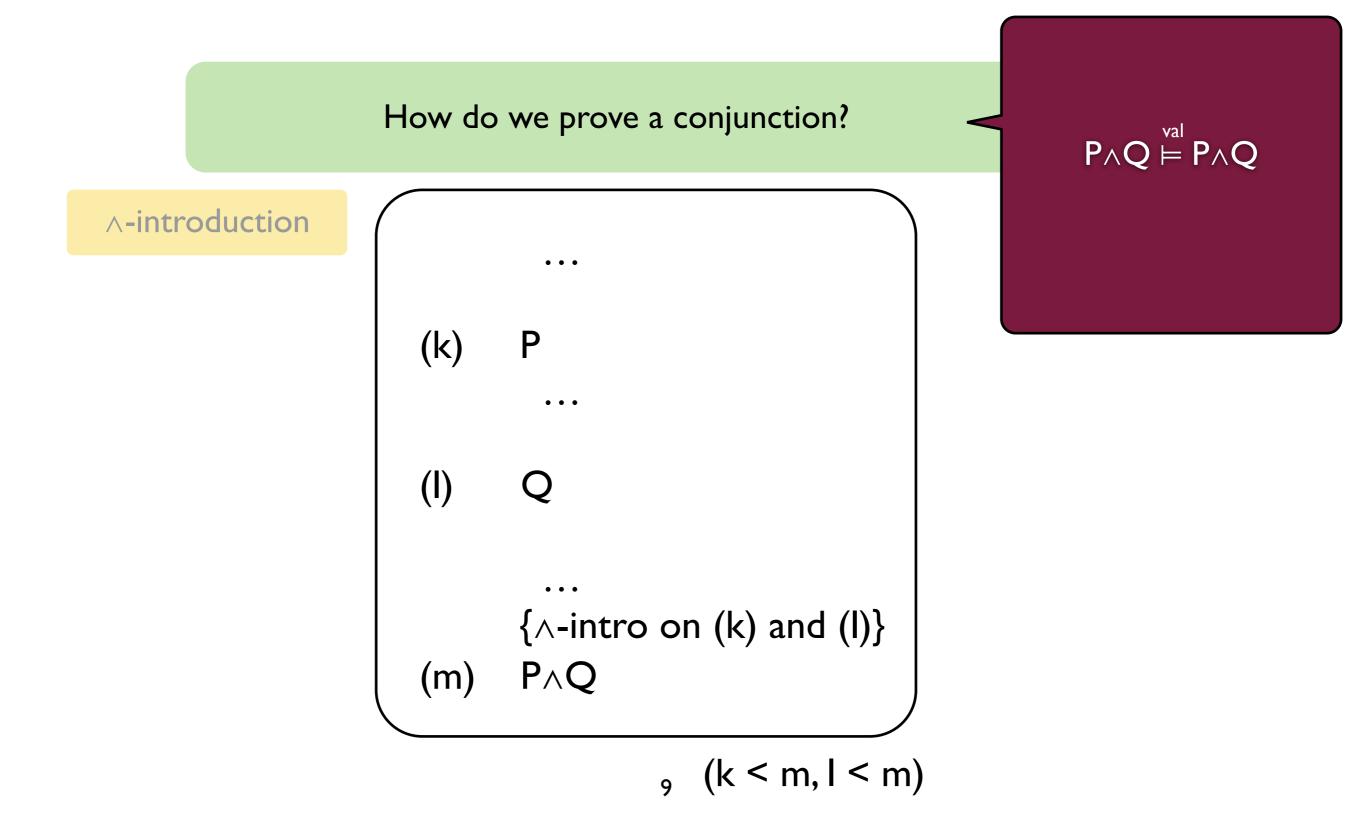
Conjunction elimination



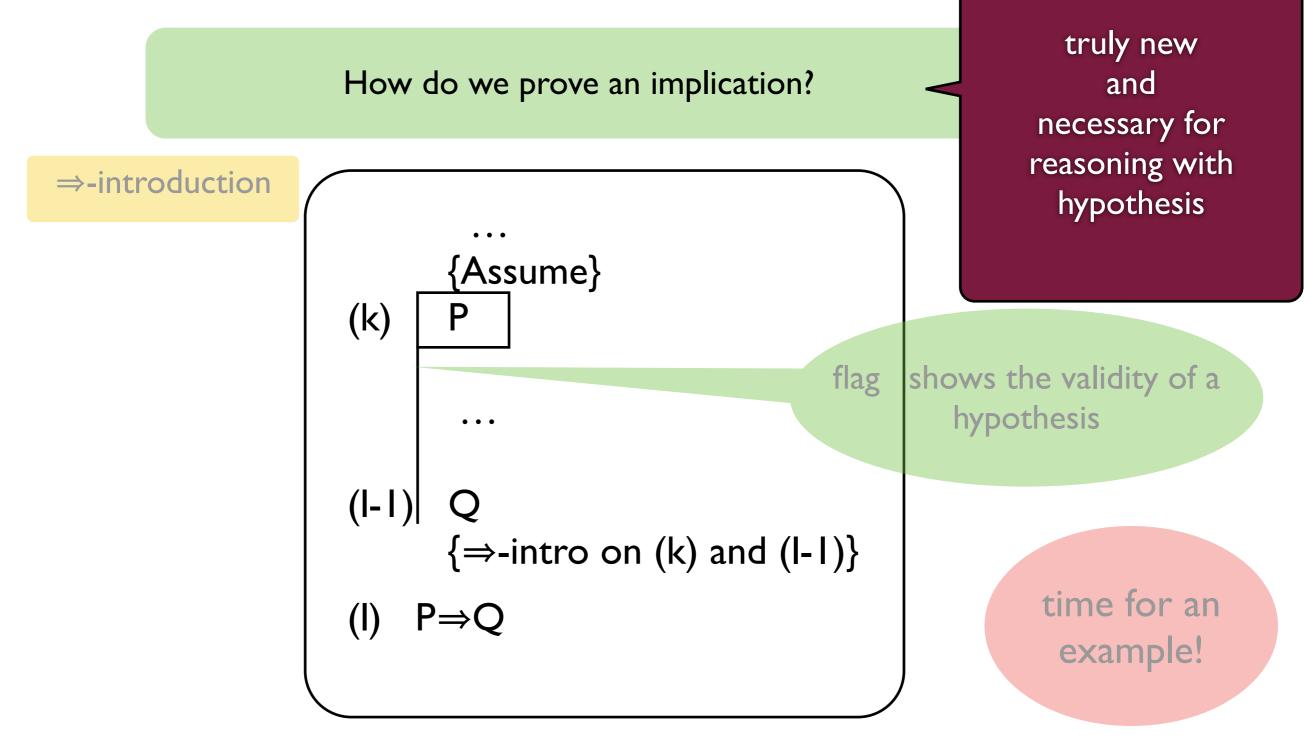
Implication elimination



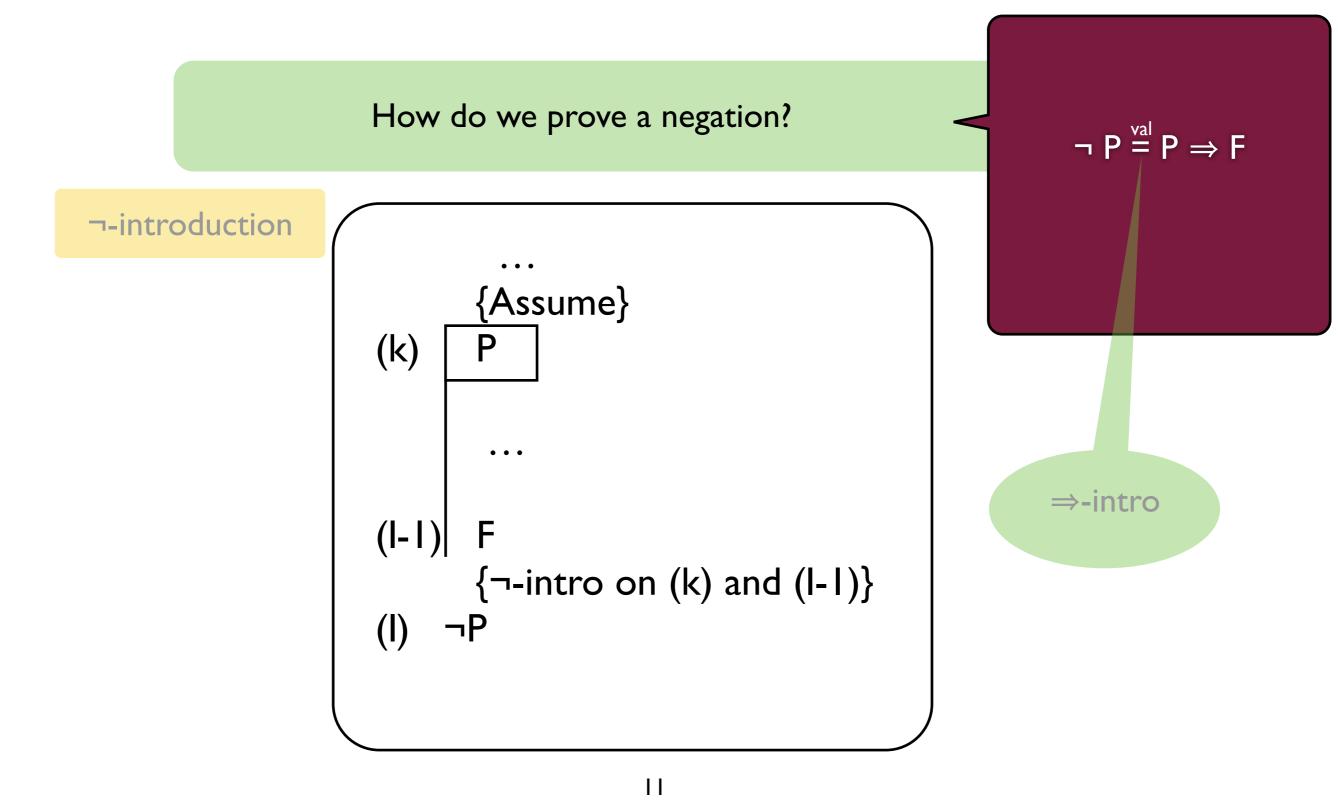
Conjunction introduction



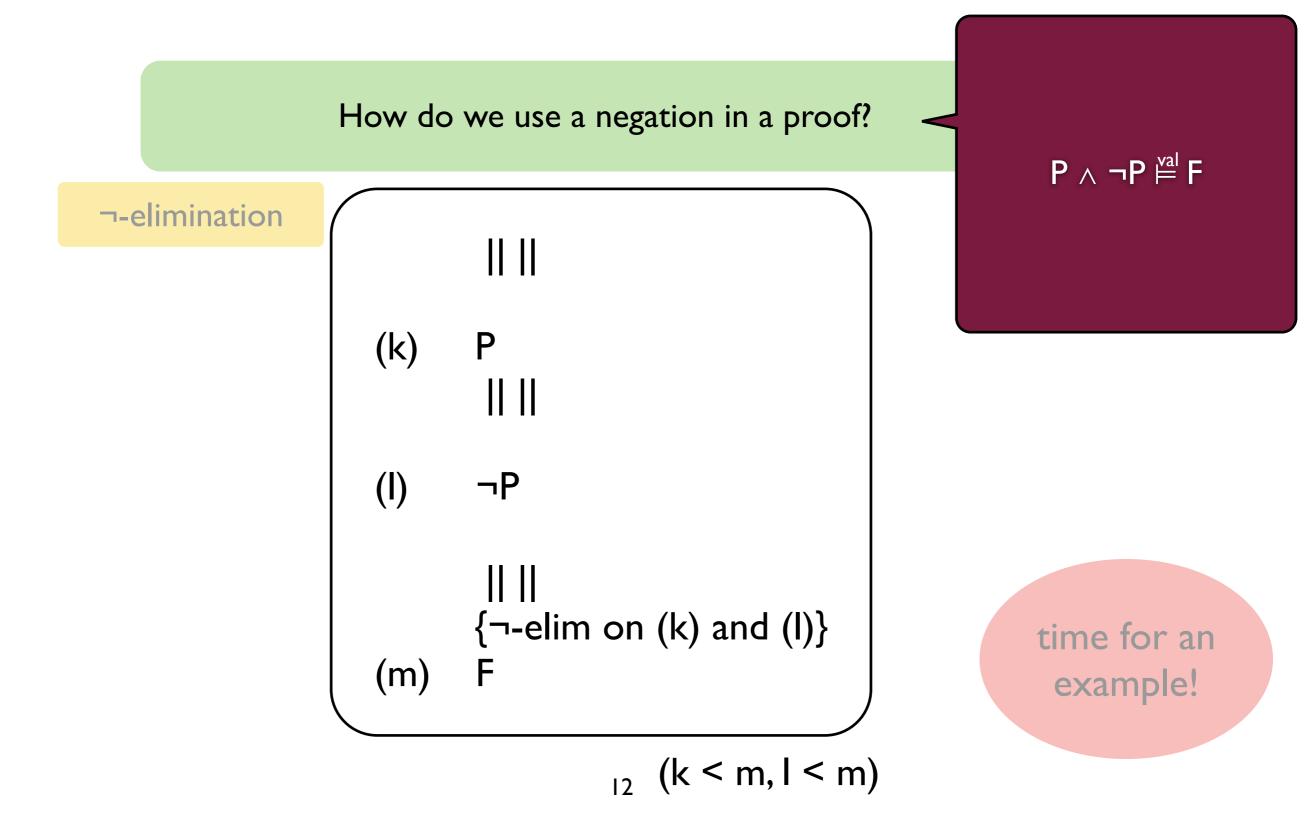
Implication introduction



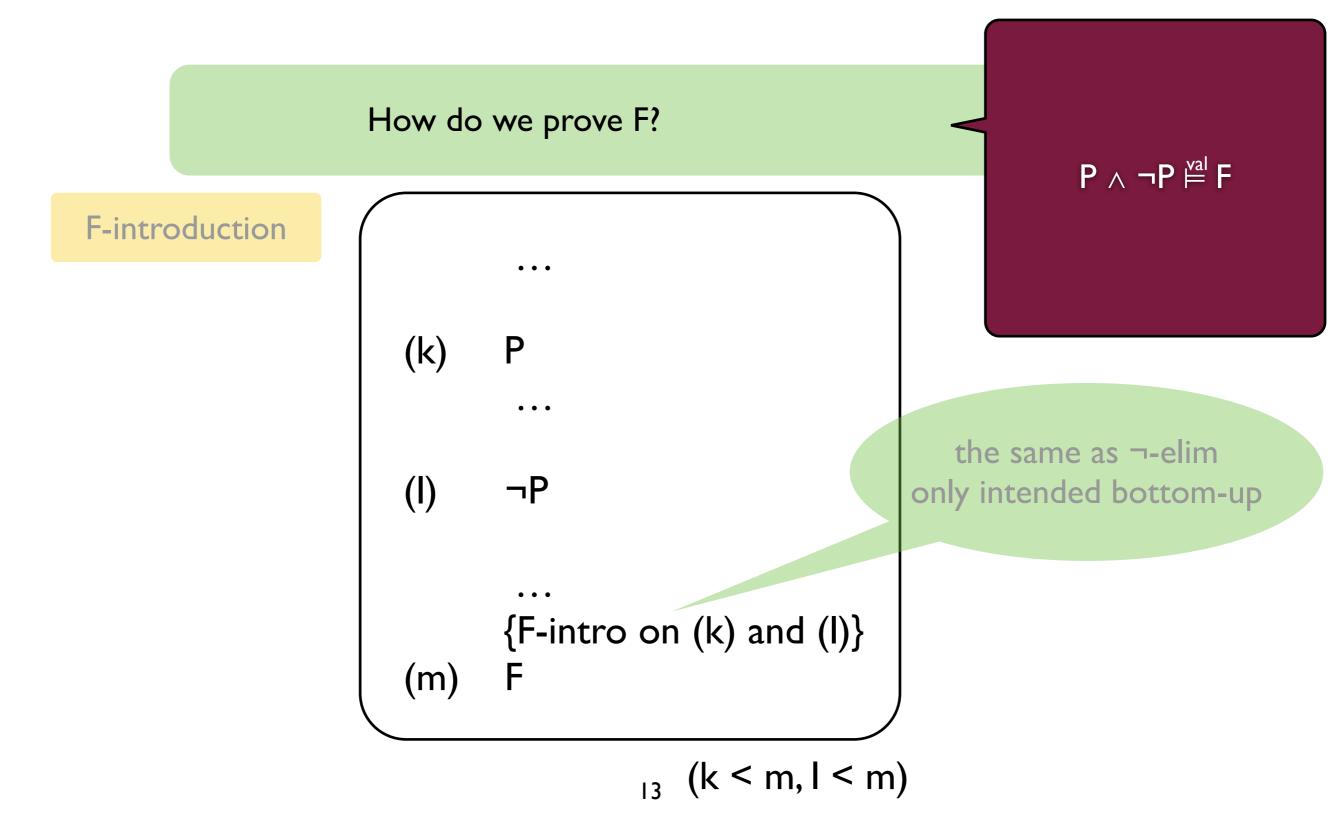
Negation introduction



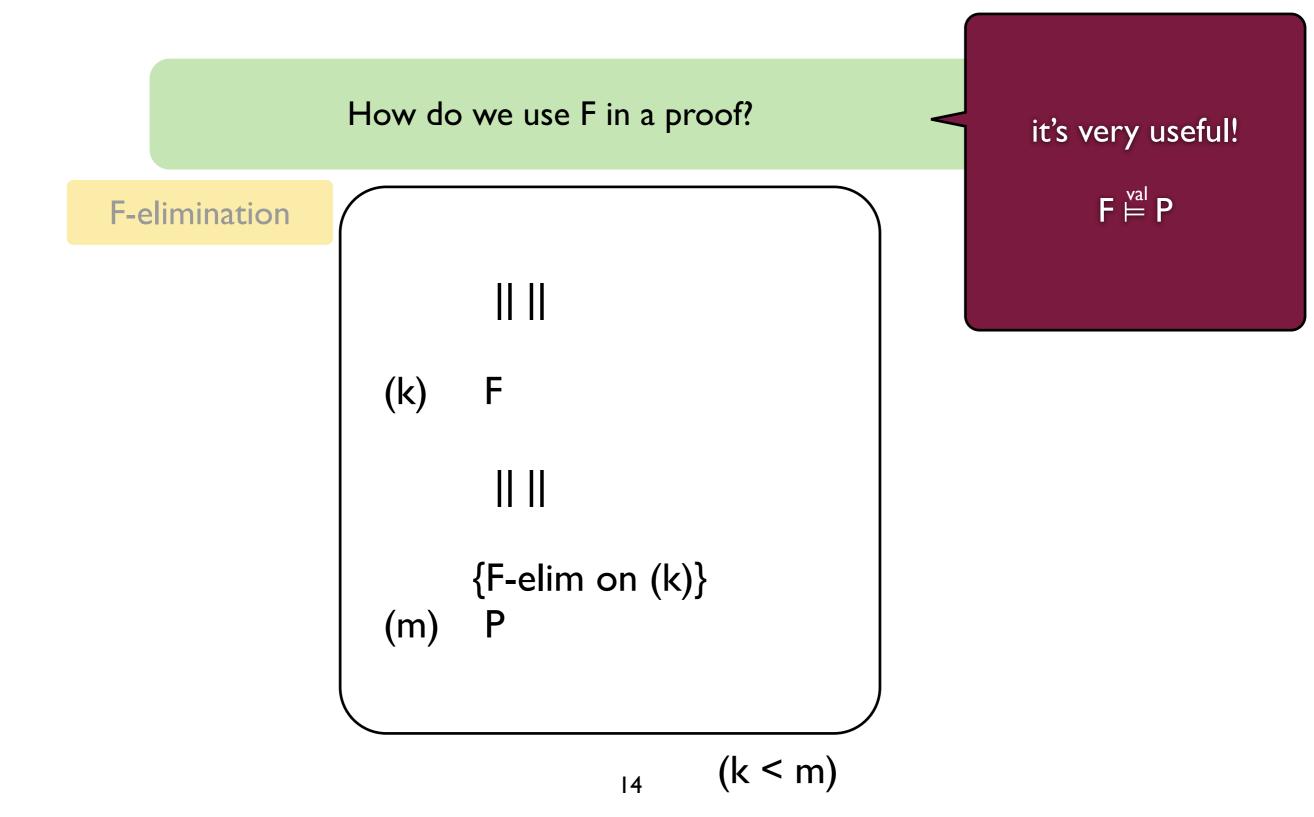
Negation elimination



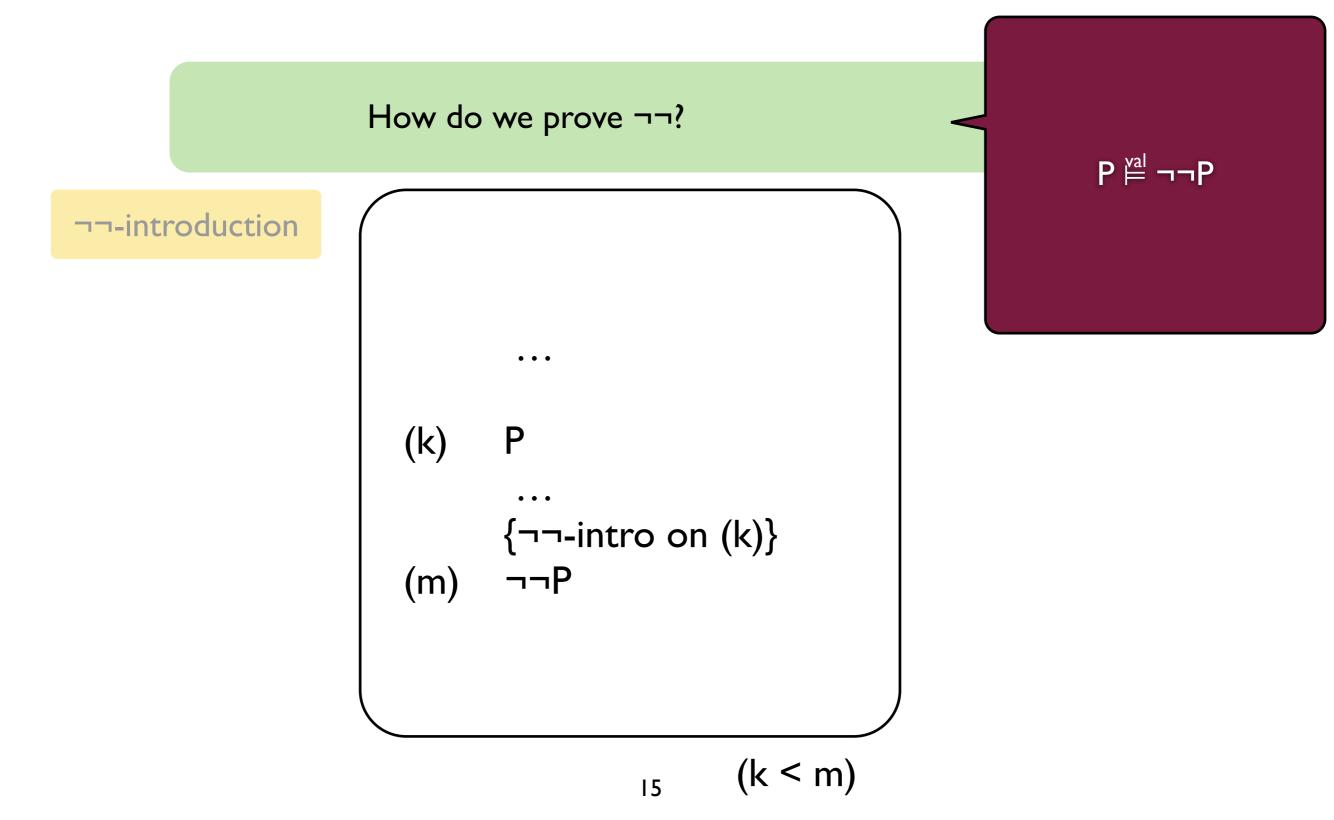
F introduction



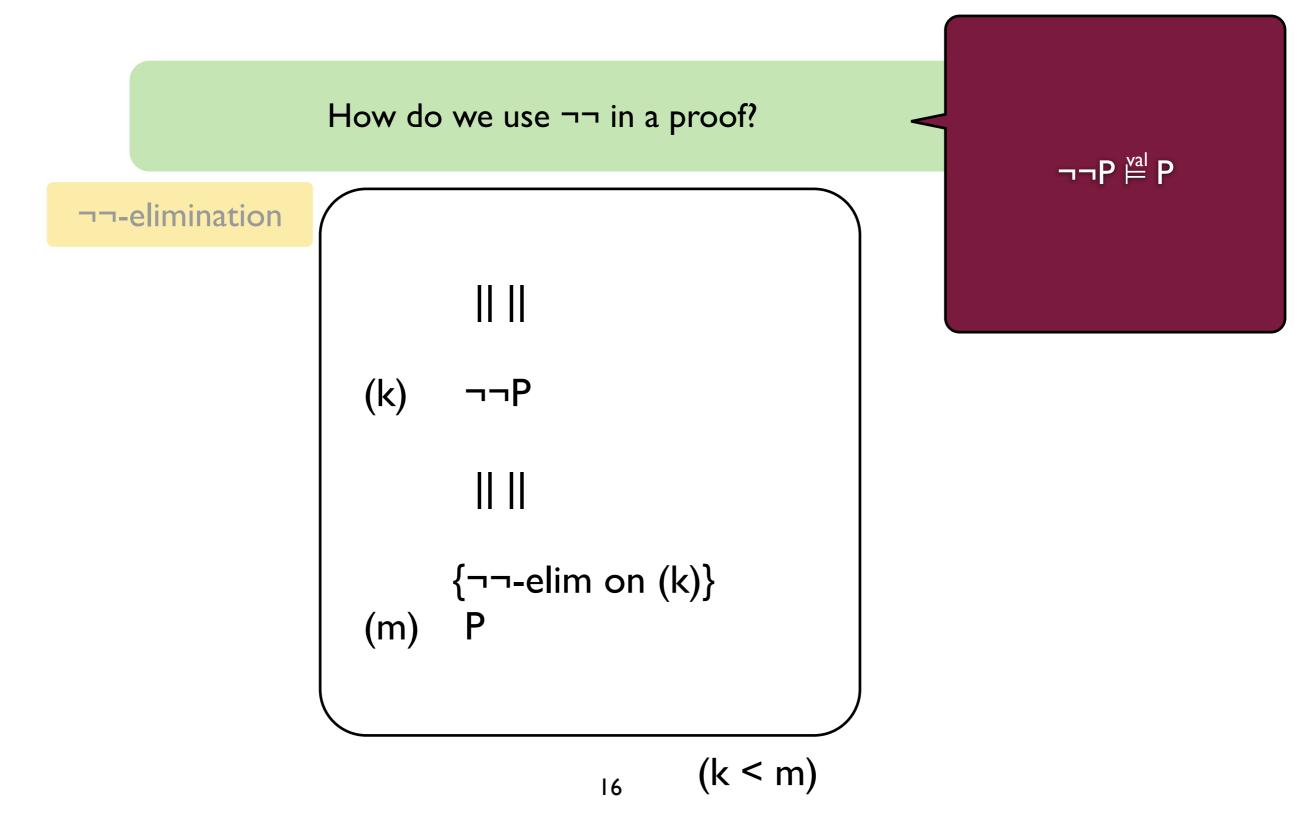
F elimination



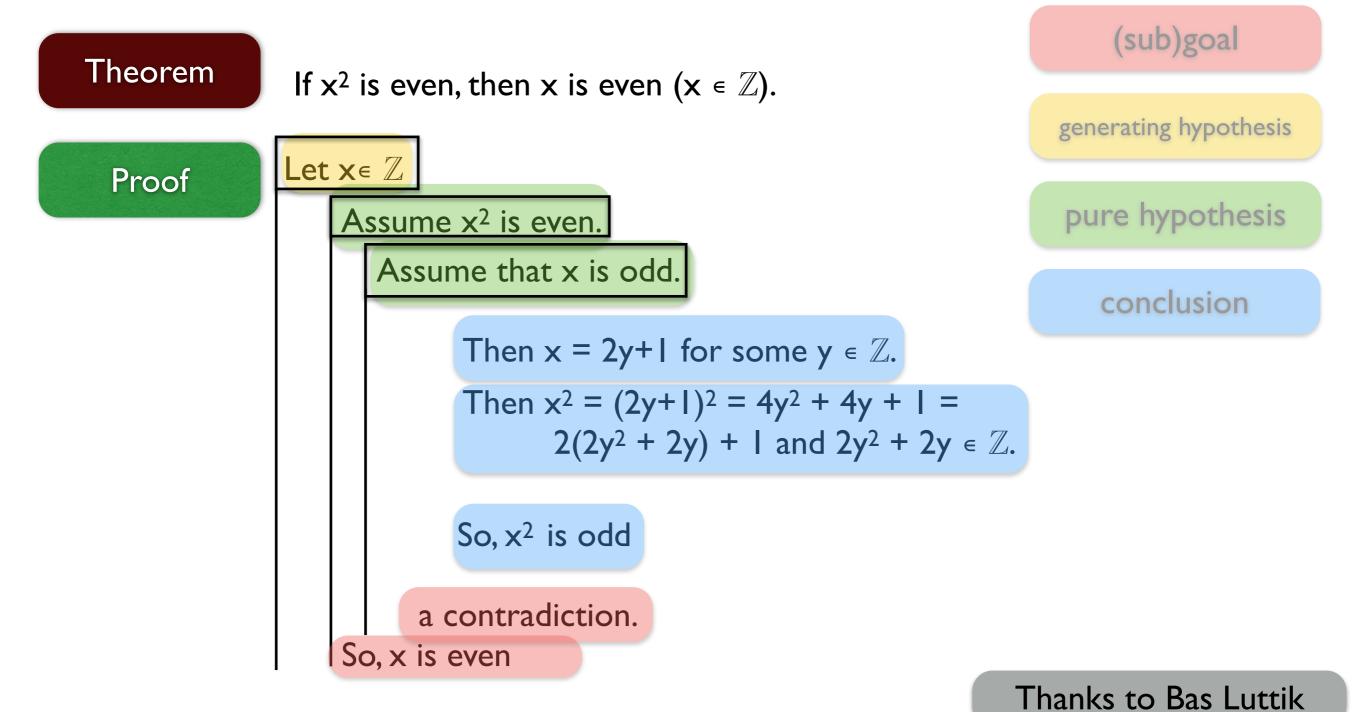
Double negation introduction



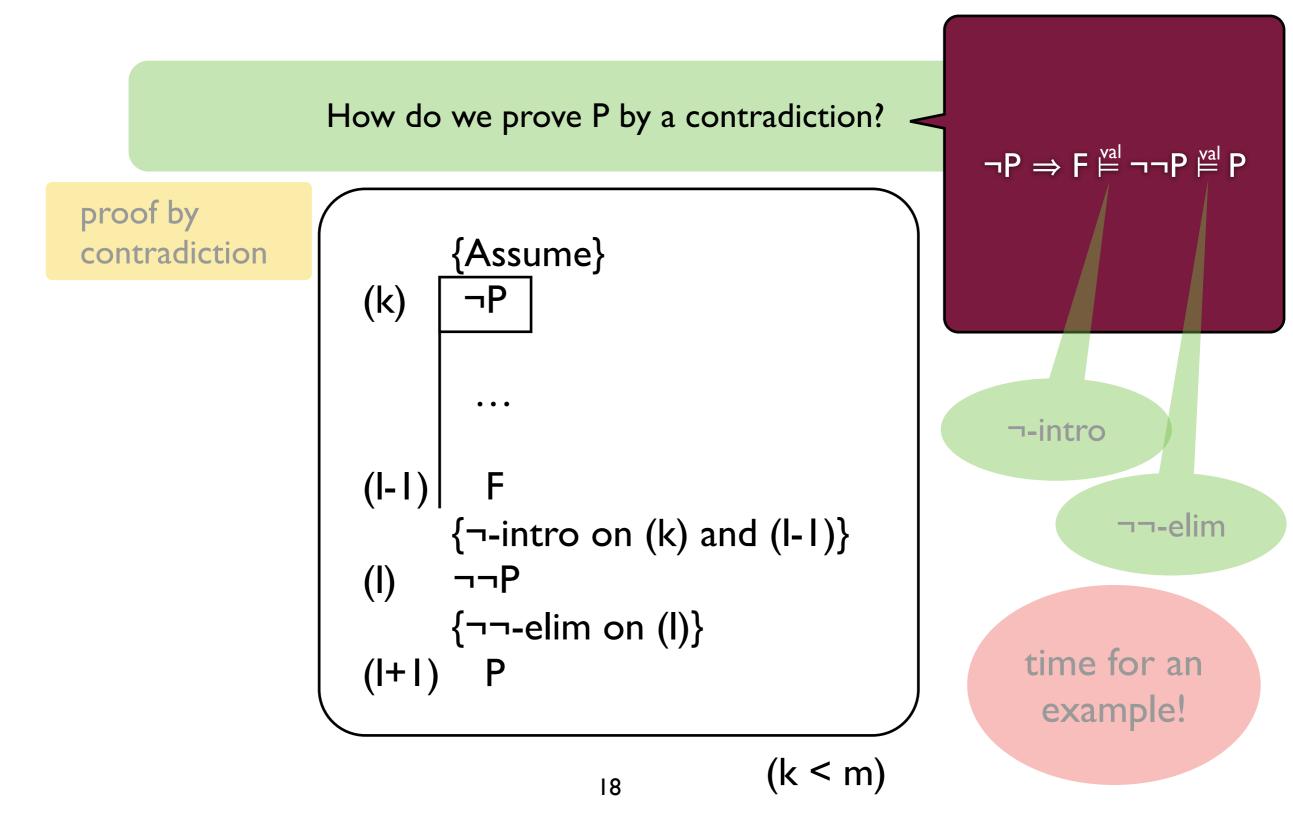
Double negation elimination



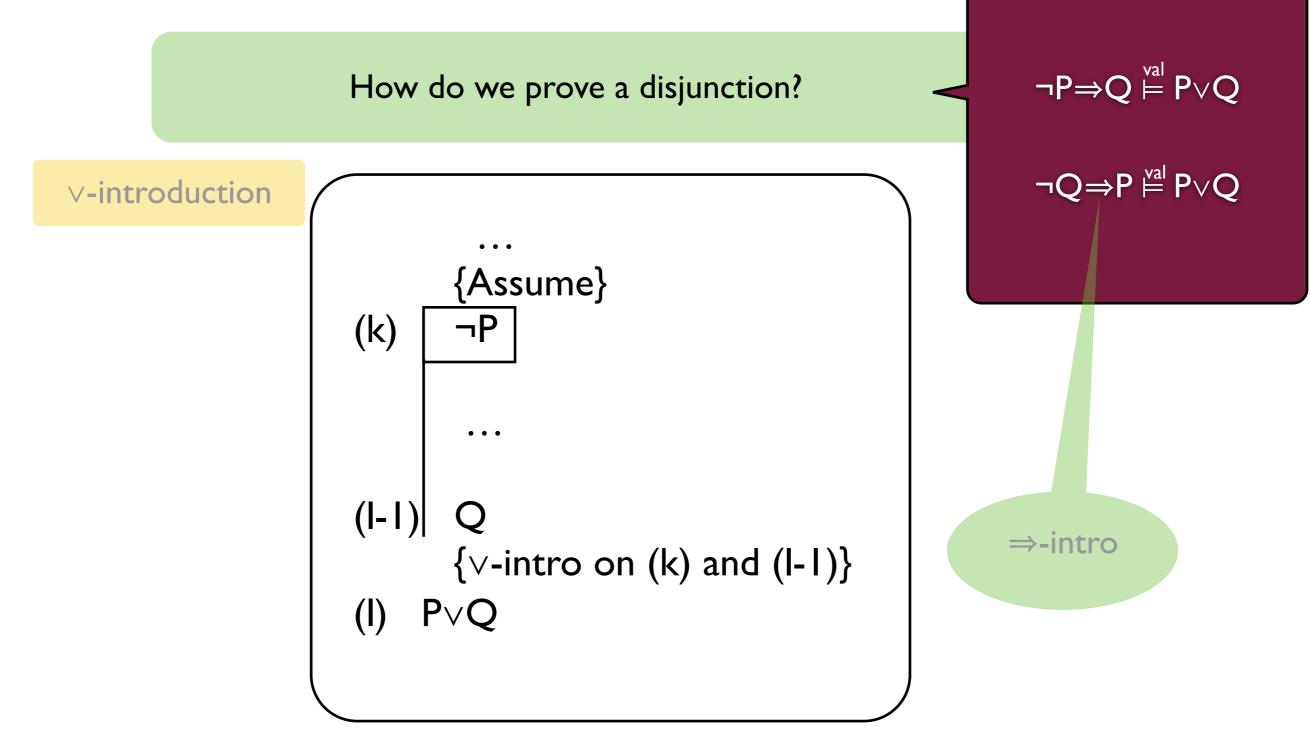
Proof by contradiction



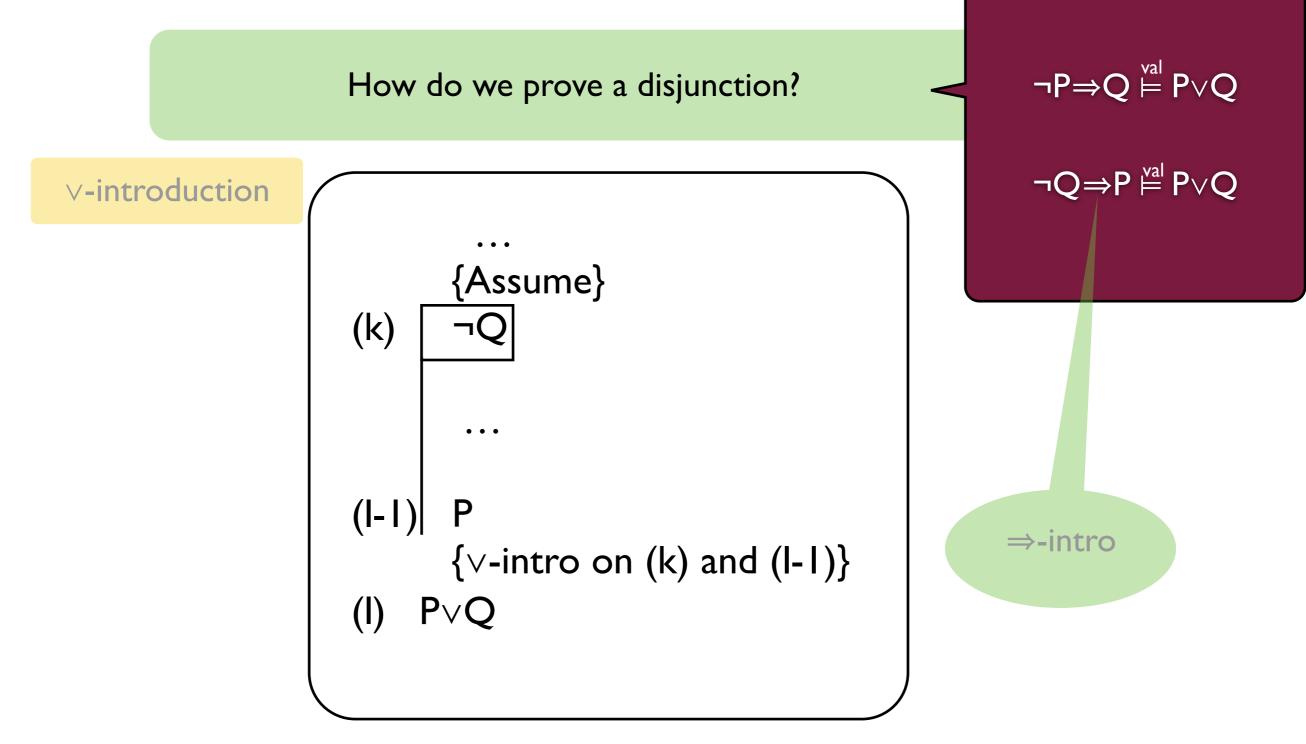
Proof by contradiction



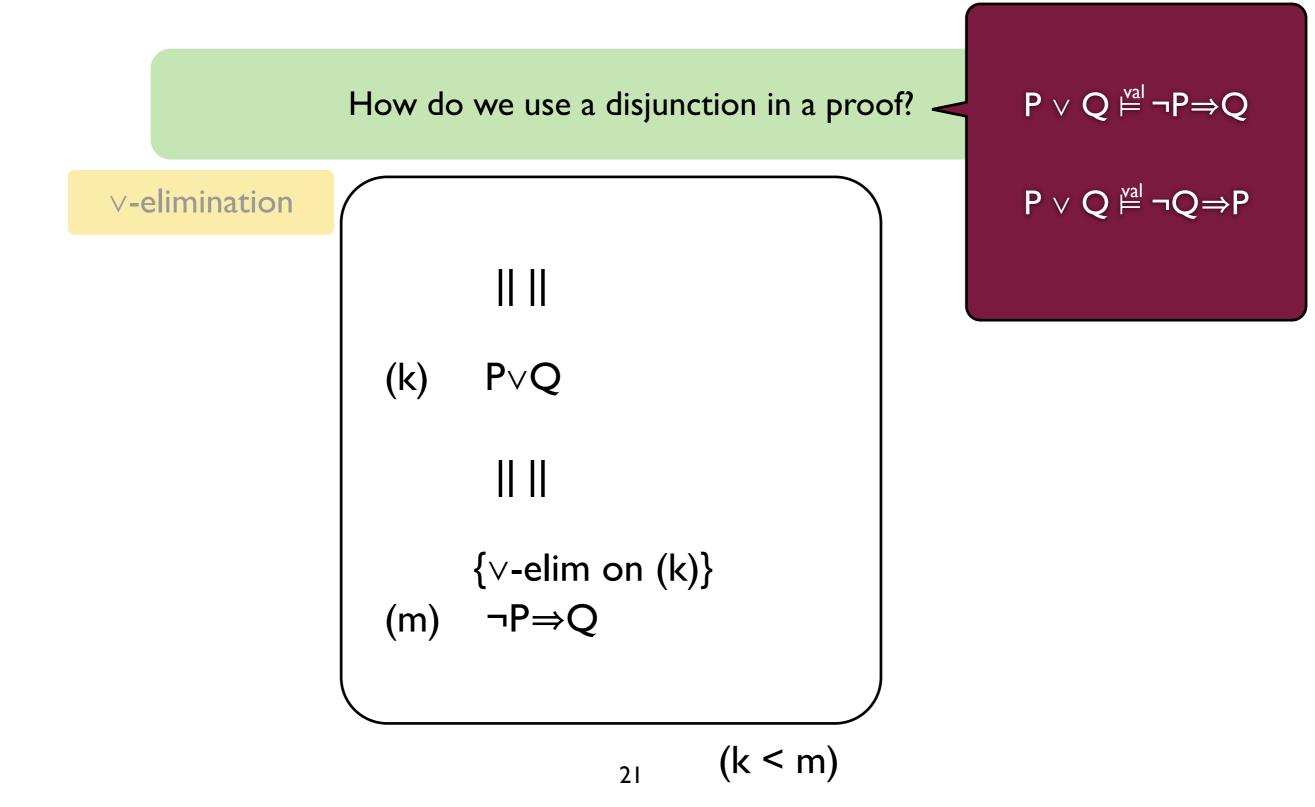
Disjunction introduction



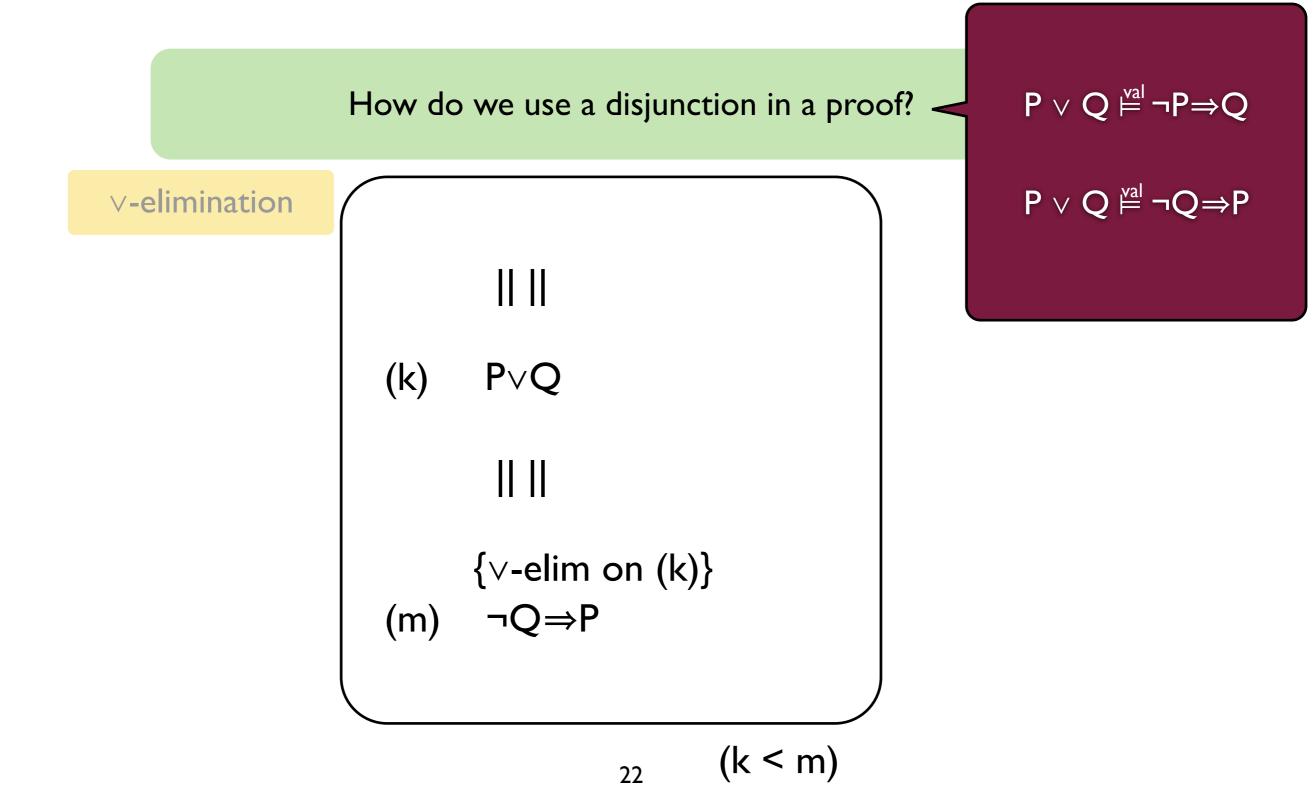
Disjunction introduction



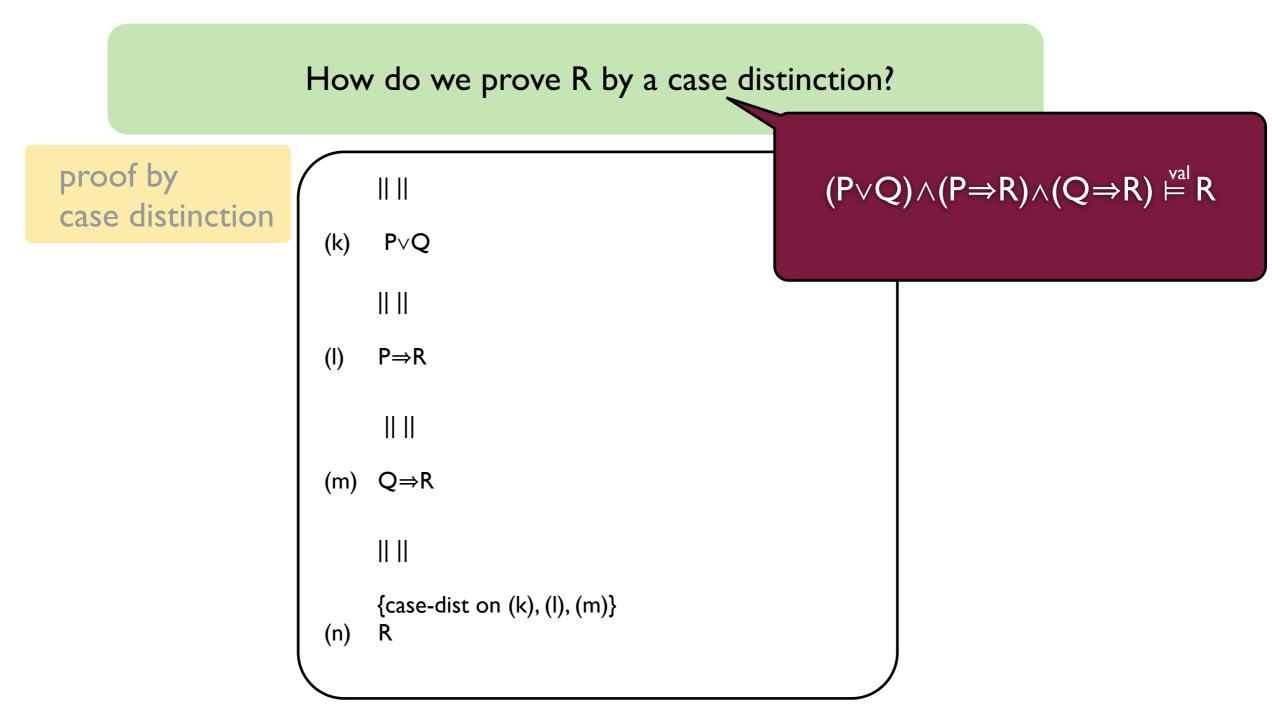
Disjunction elimination



Disjunction elimination



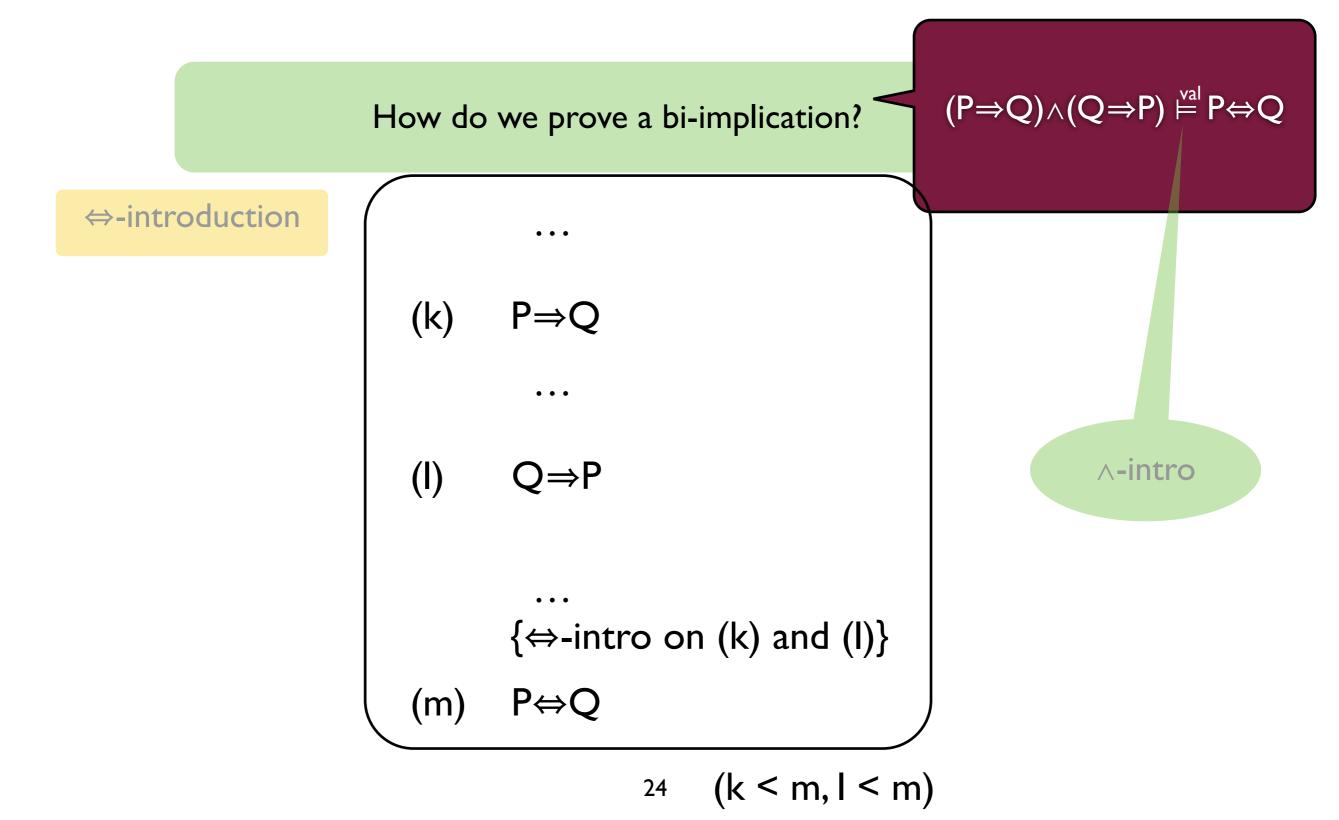
Proof by case distinction



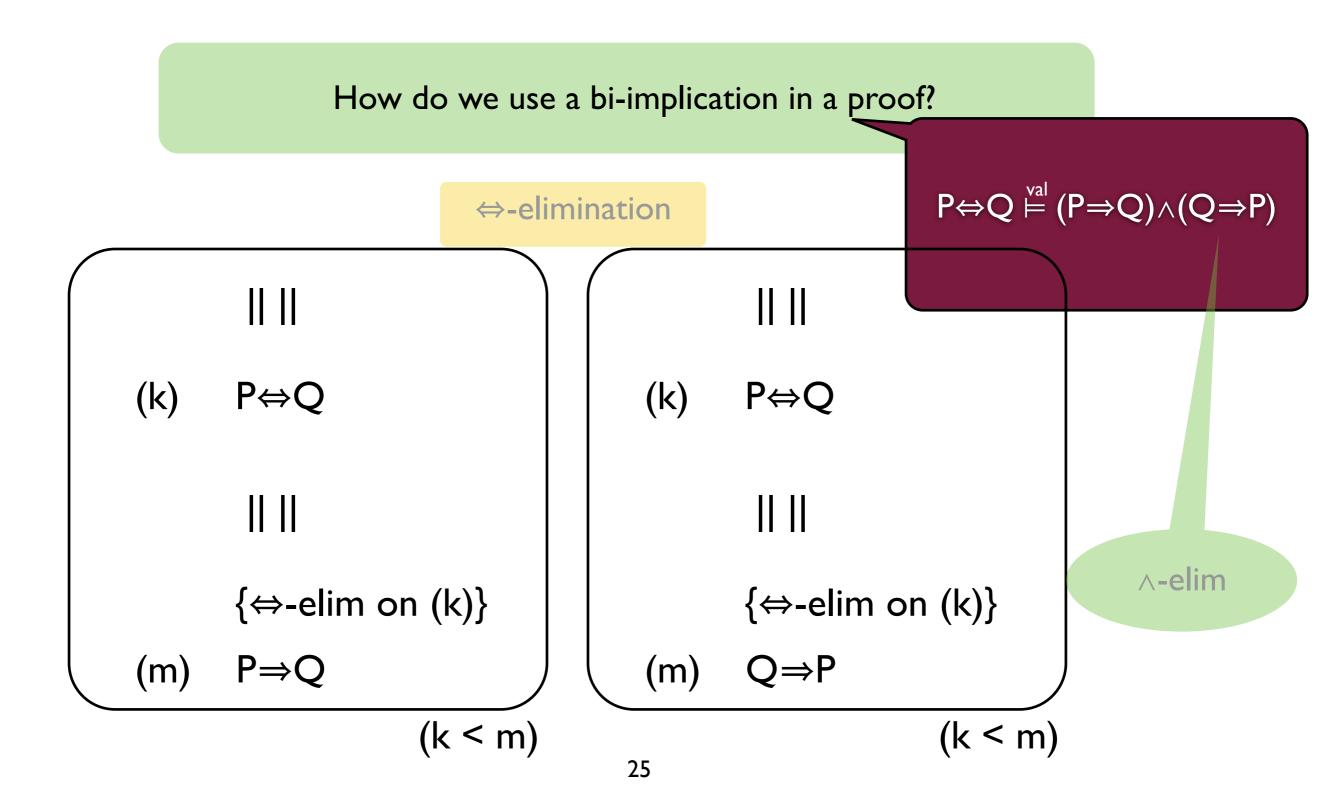
(k < n, l < n, m < n)

23

Bi-implication introduction



Bi-implication elimination



Derivations / Reasoning with quantifiers

Proving a universal quantification

To prove

 $\forall x [x \in \mathbb{Z} \land x \geq 2 : x^2 - 2x \geq 0]$

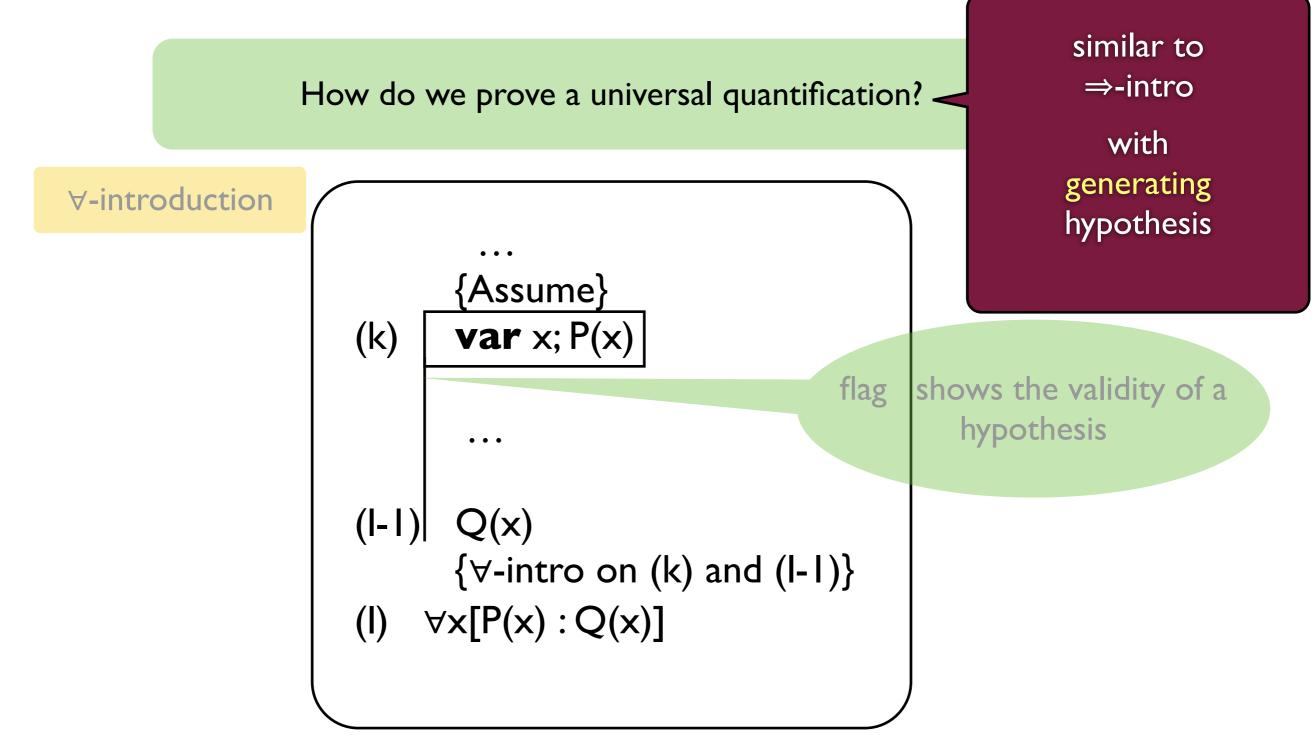
Proof

Let $x \in \mathbb{Z}$ be arbitrary and assume that $x \ge 2$.

Then, for this particular x, it holds that $x^2 - 2x = x(x-2) \ge 0$ (Why?)

Conclusion: $\forall x [x \in \mathbb{Z} \land x \ge 2 : x^2 - 2x \ge 0].$

\forall introduction



Using a universal quantification

We know

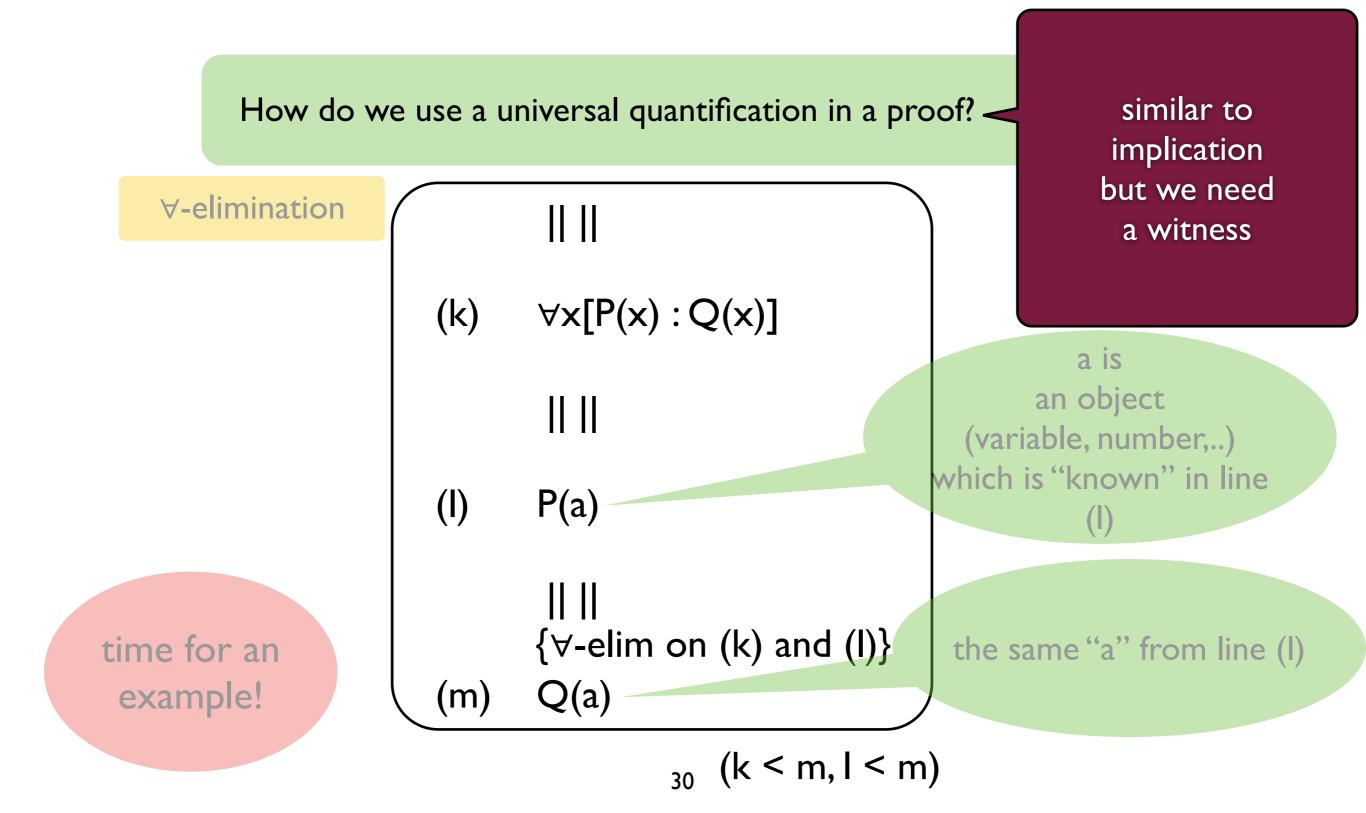
 $\forall x [x \in \mathbb{Z} \land x \geq 2 : x^2 - 2x \geq 0]$

Whenever we encounter an $a \in \mathbb{Z}$ such that $a \ge 2$,

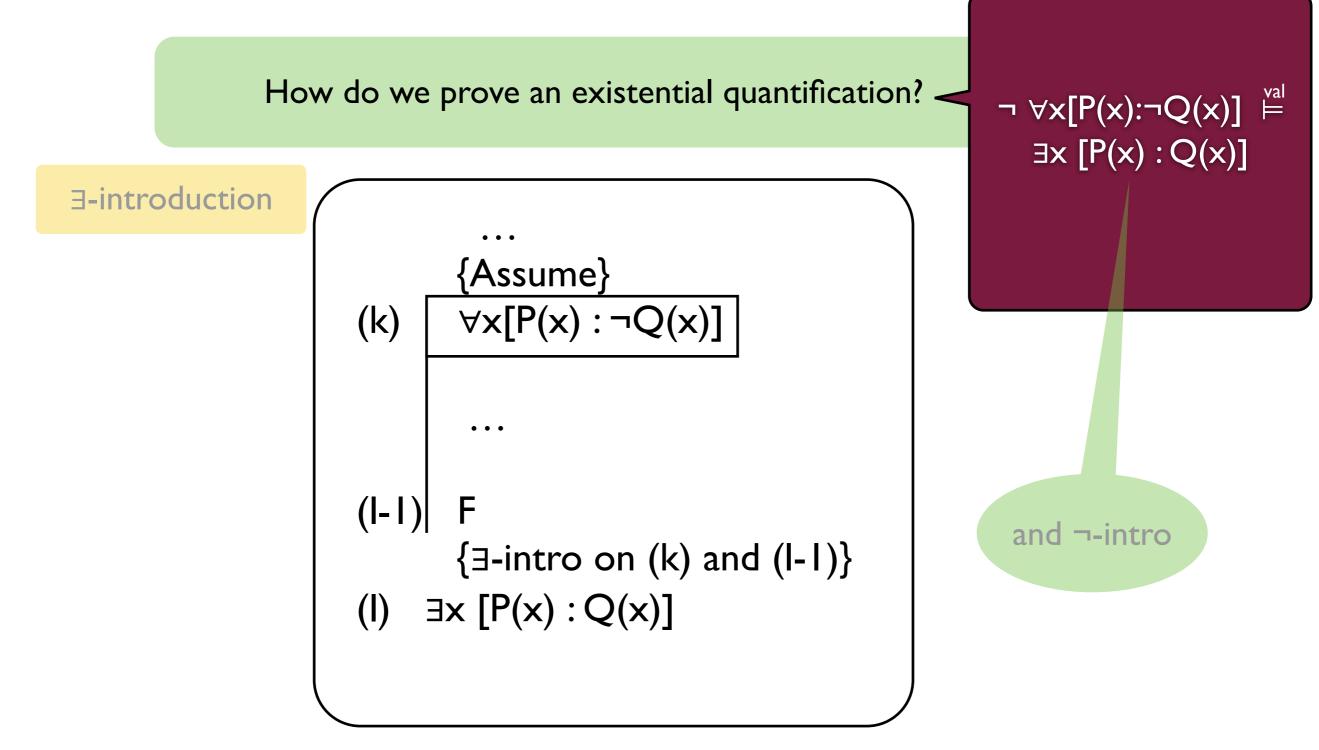
we can conclude that $a^2 - 2a \ge 0$.

For example, $(52387^2 - 2 \cdot 52387) \ge 0$ since 52387 $\in \mathbb{Z}$ and 52387 ≥ 2 .

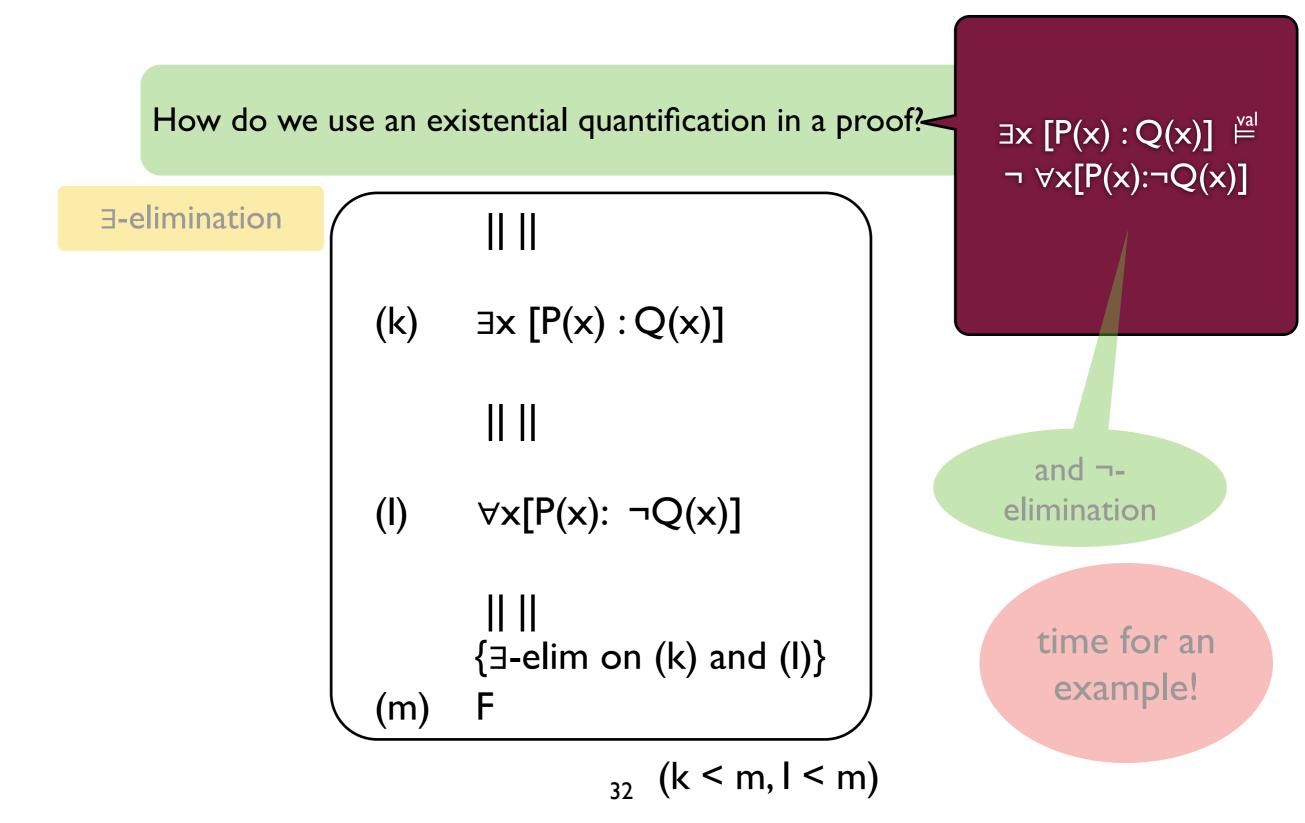
∀ elimination



∃ introduction



elimination



Proofs with 3-introduction and 3elimination are unnecessarily long and cumbersome...

There are alternatives!

Proving an existential quantification

To prove

∃x[x ∈ ℤ : x³ - 2x - 8 ≥0]

Proof

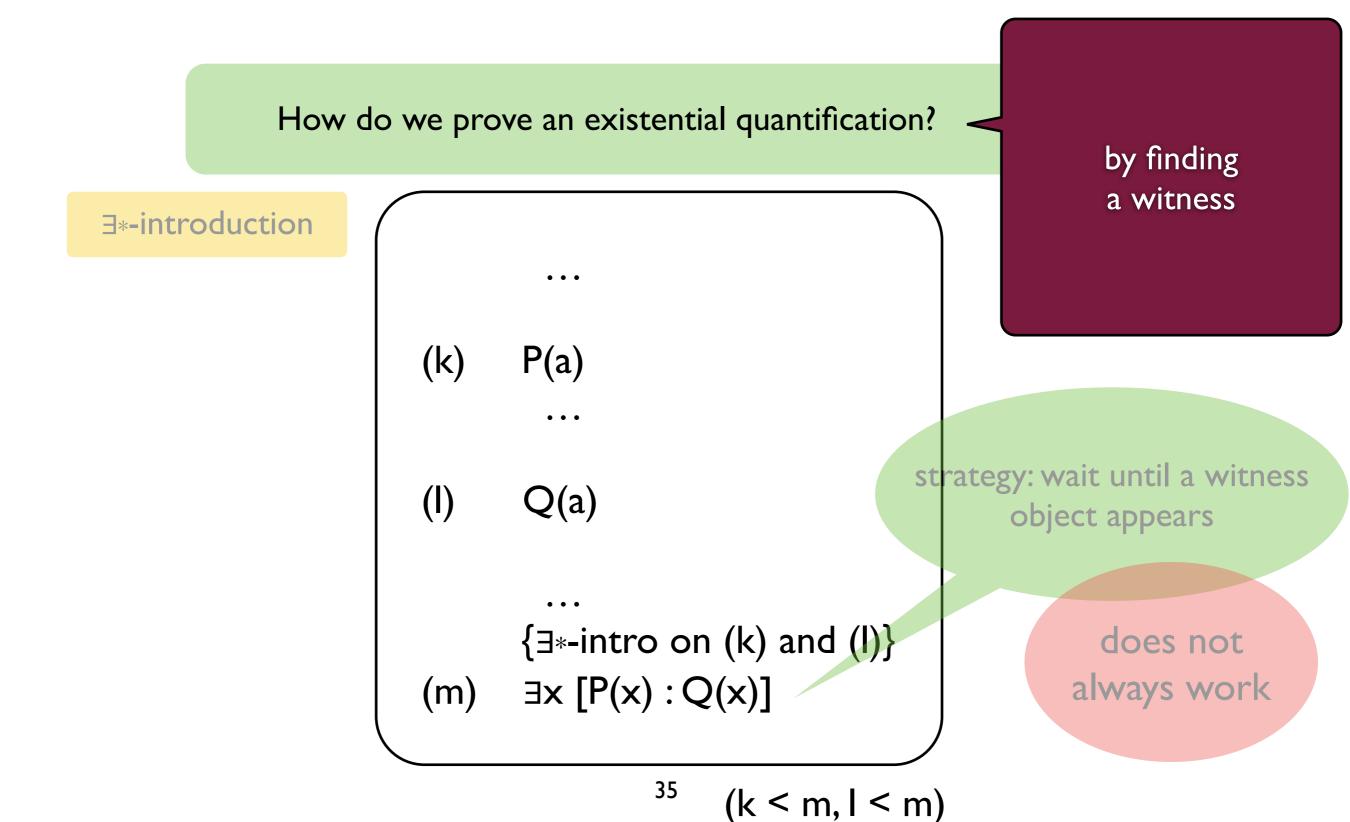
It suffices to find a witness, i.e., an $x \in \mathbb{Z}$ satisfying $x^3 - 2x - 8 \ge 0$.

x = 3 is a witness, since $3 \in \mathbb{Z}$ and $3^3 - 2 \cdot 3 - 8 = 13 \ge 0$

Conclusion: $\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \ge 0]$.

also x = 5 is a witness...

Alternative 3 introduction



Using an existential quantification

We know

$$\exists x [x \in \mathbb{R} : a - x < 0 < b - x]$$

We can declare an
$$x \in \mathbb{Z}$$
 (a witness) such that
 $a - x < 0 < b - x$
and use it further in the proof. For example:
From $a - x < 0$, we get $a < x$.
From $b - x > 0$, we get $x < b$.
Hence, $a < b$.

Alternative 3 elimination

