

Sets

- A **set** S is a collection of different objects, the elements of S
- We write $x \in S$ for 'x is an element of S'
- A set 'can' be specified by
 - (1) listing its elements, e.g. $S = \{1, 3, 7, 18\}$
 - (2) **specifying a property**, e.g. $S = \{x \mid P(x)\}$
- Sets can be **finite** e.g. $\{\clubsuit, \heartsuit\}$ or **infinite** e.g. \mathbb{N}
- The set with no elements is the **empty set**, notation \emptyset
- The 'number' of elements in a set S is the **cardinality** of S , notation $|S|$

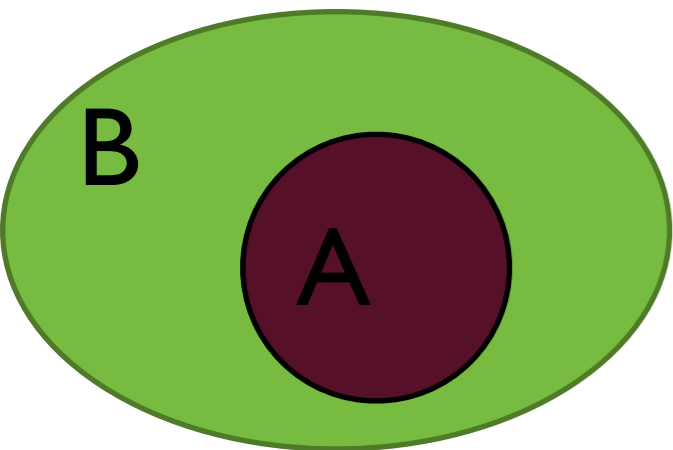
P is a proposition over x , which is true or false

Sets - properties

- All elements of a set are different
- The elements of a set are not ordered
- The same set can be specified in different ways, e.g.
 $\{1,2,3,4\}$, $\{2,3,1,4\}$, $\{i \mid i \in \mathbb{N} \text{ and } 0 < i < 5\}$

Subsets, equality

Def. $A \subseteq B$ iff all elements of A are elements of B
[iff for all a , if $a \in A$, then $a \in B$]



[iff $\forall a (a \in A \Rightarrow a \in B)$]

logical formula

quantifier

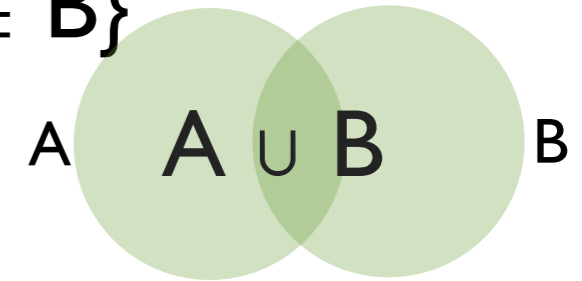
logical
connective

Def. $A = B$ iff $A \subseteq B$ and $B \subseteq A$

Def. $A \subset B$ iff $A \subseteq B$ and $A \neq B$

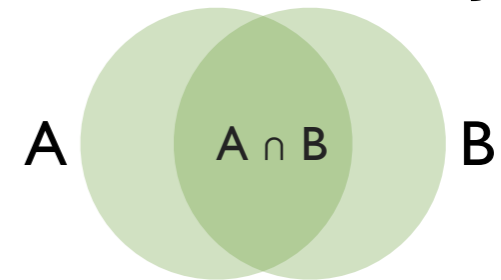
Operations on sets

Def. Union (Vereinigung) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

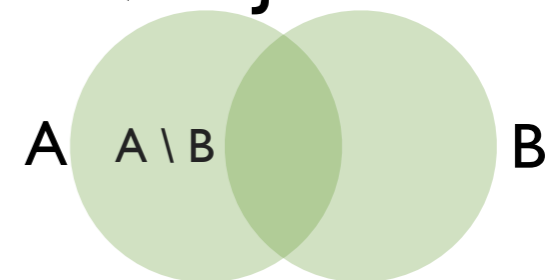


Def. Intersection (Durchschnitt) $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$

A and B are **disjoint** if $A \cap B = \emptyset$



Def. Difference (Differenz) $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



Def. Direct product (Kartesisches Produkt)

$(A \times B) \times C \neq A \times (B \times C)$

$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$

ordered pairs

Def. Powerset (Potenzmenge) $\mathcal{P}(A) = \{X \mid X \subseteq A\}$

Russell's paradox

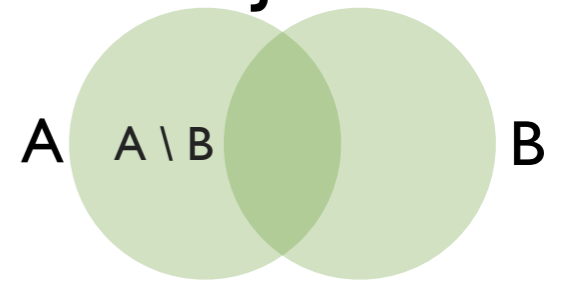
- Let P be the set of all sets that are not an element of itself
- Hence, $P = \{x \mid x \notin x\}$
- Is $P \in P$?
- **Contradiction!**

The need for a universal set U

$$S = \{x \mid x \in U \text{ and } P(x)\}$$

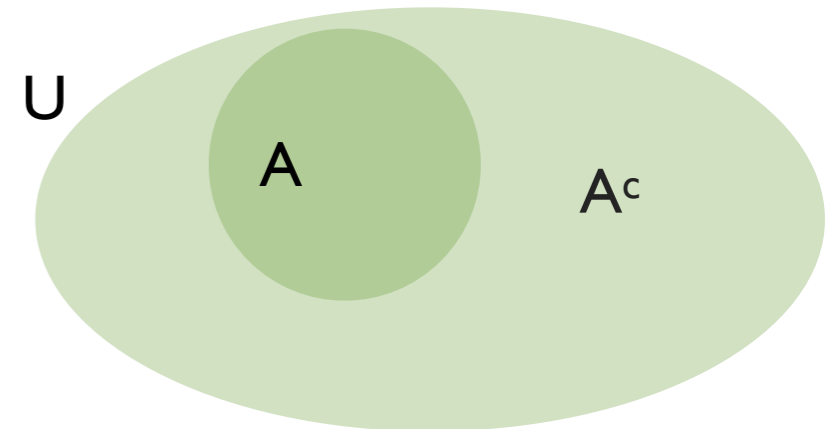
Operations on sets

Def. Difference (Differenz) $A \setminus B = \{x \mid x \in A \text{ and } x \notin B\}$



Given a universal set U

Def. Complement (Komplement) $A^c = \{x \mid x \in U \text{ and } x \notin A\}$



Hence $A^c = U \setminus A$

Properties of sets

1. $\emptyset \subseteq X$

2. If $X \subseteq Y$ and $Y \subseteq Z$, then $X \subseteq Z$

3. $X \cap Y \subseteq X$, $X \cap Y \subseteq Y$

4. $X \subseteq X \cup Y$, $Y \subseteq X \cup Y$

5. If $X' \subseteq Y'$ and $X'' \subseteq Y''$, then $X' \cap X'' \subseteq Y' \cap Y''$

6. If $X' \subseteq Y'$ and $X'' \subseteq Y''$, then $X' \cup X'' \subseteq Y' \cup Y''$

7. $X \cap Y = X$ iff $X \subseteq Y$

8. $X \cap X = X$ (idempotence)

9. $X \cup X = X$ (idempotence)

10. $X \cap \emptyset = \emptyset$

Properties of sets

$$11. X \cup \emptyset = X$$

$$12. X \cap Y = Y \cap X \text{ (commutativity)}$$

$$13. X \cup Y = Y \cup X \text{ (commutativity)}$$

$$14. X \cap (Y \cap Z) = (X \cap Y) \cap Z \text{ (associativity)}$$

$$15. X \cup (Y \cup Z) = (X \cup Y) \cup Z \text{ (associativity)}$$

$$16. X \cap (X \cup Y) = X \text{ (absorption)}$$

$$17. X \cup (X \cap Y) = X \text{ (absorption)}$$

$$18. X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z) \text{ (distributivity)}$$

$$19. X \cup (Y \cap Z) = (X \cup Y) \cap (X \cup Z) \text{ (distributivity)}$$

$$20. X \setminus Y \subseteq X$$

Properties of sets

$$21. (X \setminus Y) \cap Y = \emptyset$$

$$22. X \cup Y = X \cup (Y \setminus X)$$

$$23. X \setminus X = \emptyset$$

$$24. X \setminus \emptyset = X$$

$$25. \emptyset \setminus X = \emptyset$$

$$26. \text{If } X \subseteq Y, \text{ then } X \setminus Y = \emptyset$$

$$27. (X^c)^c = X$$

$$28. (X \cap Y)^c = X^c \cup Y^c \quad (\text{De Morgan})$$

$$29. (X \cup Y)^c = X^c \cap Y^c \quad (\text{De Morgan})$$

$$30. X \times \emptyset = \emptyset$$

$$31. \emptyset \times X = \emptyset$$

$$32. \text{If } X \subseteq Y, \text{ then } \mathcal{P}(X) \subseteq \mathcal{P}(Y) \quad 9$$