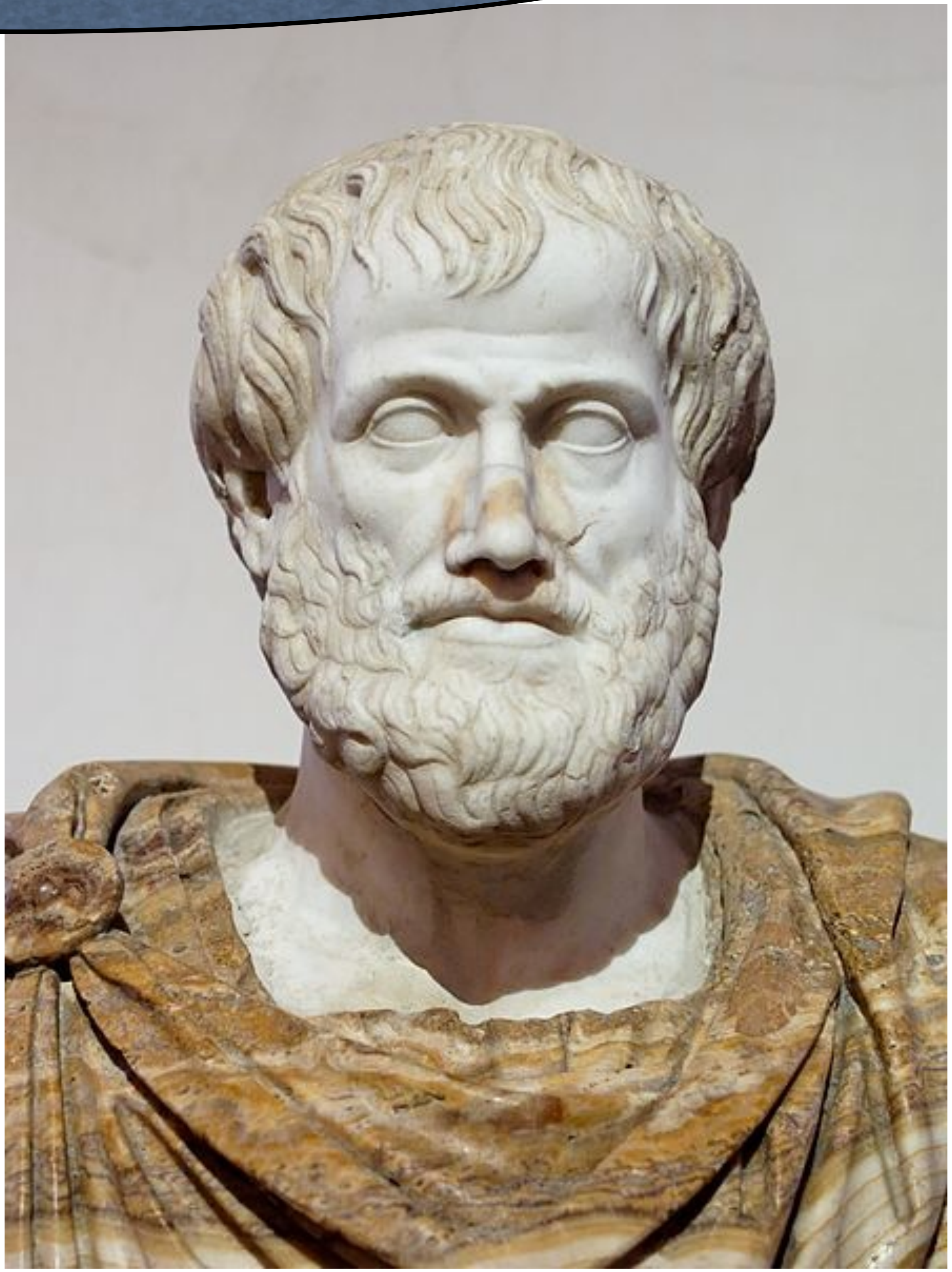
Logic

Logic = study of correct reasoning

In the beginning



Aristotle +/- 350 B.C.

Organon

19 syllogisms

Barbara syllogism

only later called so, in the Middle Ages

All K's are L's
All L's are M's

All K's are M's

from the two premises

one can **always** conclude the conclusion

independent of what the parameters K,L,M are

Logic (Logos, Greek for word, understanding, reason) deals with general reasoning laws in the form of formulas with parameters.

Propositions

Def. A proposition (Aussage) is a grammatically correct sentence that is either true or false.

logic deals with patterns! what matters are not particular propositions but the way in which (abstract) propositions are combined and what follows from them

Connectives

$\wedge$ for "and"

$\vee$ for "or"

$\neg$ for "not"

$\Rightarrow$ for "if .. then" or "implies"

$\Leftrightarrow$ for "if and only if"

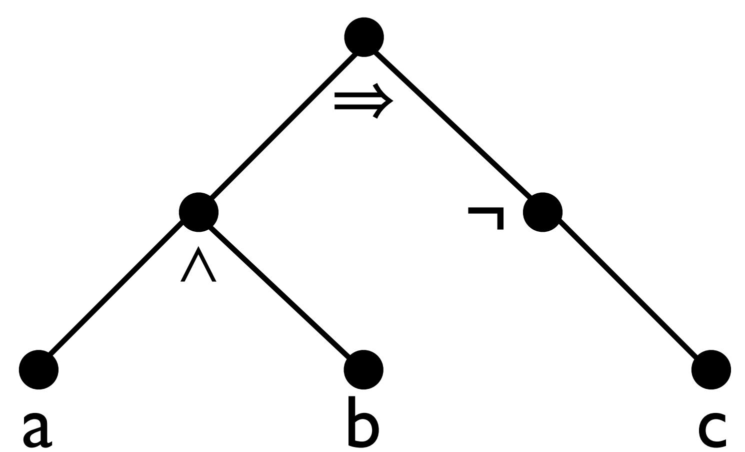
Abstract propositions

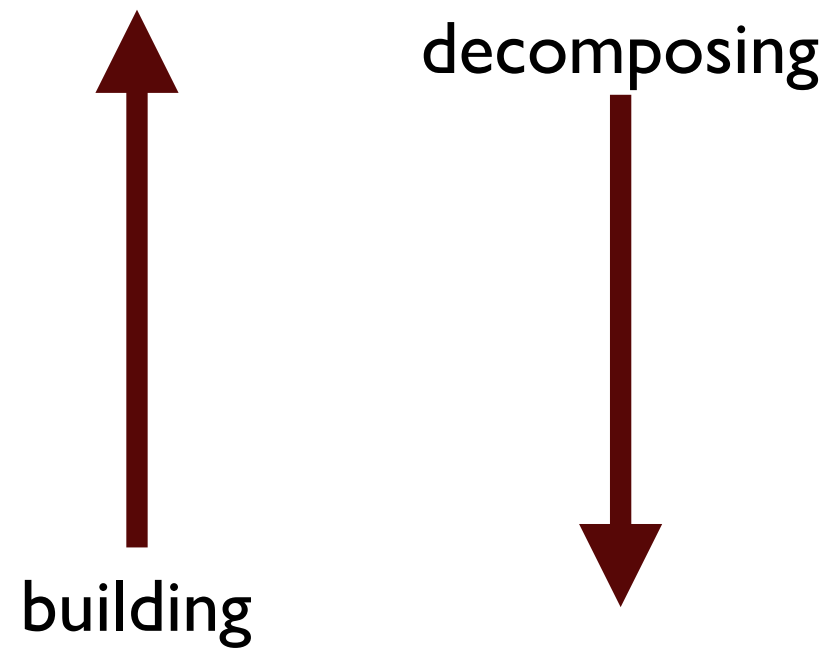
Definition

Basis	Propositional variables are abstract propositions.
Step (Case 1)	If P is an abstract proposition, then so is $(\neg P)$.
Step (Case 2)	If P and Q are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

a recursive/inductive definition

…and their structure





the tree of
$((a \wedge b) \Rightarrow (\neg c))$

tree representation
(no need of
parenthesis)

Dropping parenthesis

$\neg$

$\wedge \quad \vee$

$\Rightarrow$

$\Leftrightarrow$

priority schema
(top binds the most)

decreasing priority

increasing priority

Example: $((a \wedge b) \Rightarrow (\neg c))$

becomes

$a \wedge b \Rightarrow \neg c$

Truth tables

Conjunction

P	Q	P∧Q
0	0	0
0	1	0
1	0	0
1	1	1

only true when both P and Q are true

Truth tables

Disjunction

P	Q	P∨Q
0	0	0
0	1	1
1	0	1
1	1	1

true when either P or Q or both are true

Truth tables

Negation

unary connective

P	¬P
0	1
1	0

true when P is false

Truth tables

Implication

needs more attention

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

only false when P is true and Q is false

Truth tables

Bi-implication

$P \Leftrightarrow Q$

is $(P \Rightarrow Q) \land (Q \Rightarrow P)$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

true when P and Q have the same truth value

Truth-functions

Def. A truth-function or Boolean function is a function

$$f: \{0,1\}^n \longrightarrow \{0,1\}$$

$a_1, .. a_n$ are the variables in P (and more) ordered in a sequence

Property: Every abstract proposition $P(a_1,..,a_n)$ induces a truth-function.

by its inductive structure, using the truth tables

Notation in the book…

$$\begin{array}{l} \quad a, b \\ \left\{ \begin{array}{l} (0,0) \longmapsto 0 \\ (0,1) \longmapsto 1 \\ (1,0) \longmapsto 0 \\ (1,1) \longmapsto 1 \end{array} \right. \end{array}$$

P(a,b): $(a \wedge b) \vee b$

Truth-functions

$a_1, .. a_n$ are the variables in P (and more) ordered in a sequence

Property: Every abstract proposition $P(a_1,..,a_n)$ with ordered and specified variables induces a truth-function.

Note:

The sequence of specified variables matters!

$P(a,b,c)$: $(a \wedge b) \vee b$

induces

a, b, c		
(0,0,0)	$\longmapsto$	0
(0,0,1)	$\longmapsto$	0
(0,1,0)	$\longmapsto$	1
(0,1,1)	$\longmapsto$	1
(1,0,0)	$\longmapsto$	0
(1,0,1)	$\longmapsto$	0
(1,1,0)	$\longmapsto$	1
(1,1,1)	$\longmapsto$	1

Equivalence of propositions

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{val}{=} Q$, iff they induce the same truth-function

on any sequence containing their common variables

Property: The relation $\stackrel{val}{=}$ is an equivalence on the set of all abstract propositions

i.e., for all abstract propositions P, Q, R,
(1) $P \stackrel{val}{=} P$; (2) if $P \stackrel{val}{=} Q$, then $Q \stackrel{val}{=} P$; and
(3) if $P \stackrel{val}{=} Q$ and $Q \stackrel{val}{=} R$, then $P \stackrel{val}{=} R$

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0				
0	1				
1	0				
1	1				

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1			
0	1	1			
1	0	0			
1	1	0			

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1		
0	1	1	0		
1	0	0	1		
1	1	0	0		

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1	0	
0	1	1	0	0	
1	0	0	1	0	
1	1	0	0	0	

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1	0	0
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0	1	1	0	0
0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

Their truth values are the same, so they are equivalent

$$b \wedge \neg b \stackrel{val}{=} c \wedge \neg c$$

Tautologies and contradictions

Def. An abstract proposition P is a tautology iff its truth-function is constant 1.

all tautologies are equivalent

Def. An abstract proposition P is a contradiction iff its truth-function is constant 0.

all contradictions are equivalent

but not all contingencies!

Def. An abstract proposition P is a contingency iff it is neither a tautology nor a contradiction.

Abstract propositions

Definition

Basis (Case 1) T and F are abstract propositions.
Basis (Case 2) Propositional variables are abstract propositions.

Step (Case 1) If P is an abstract proposition, then so is $(\neg P)$.
Step (Case 2) If P and Q are abstract propositions, then so are
$(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

a recursive/inductive definition

Propositional Logic
Standard Equivalences

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

$$P \Rightarrow Q \stackrel{val}{\neq} Q \Rightarrow P$$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0	1	1	0

Commutativity and Associativity

Commutativity

$$P \wedge Q \stackrel{val}{=} Q \wedge P$$

$$P \vee Q \stackrel{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \stackrel{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$
$$P \vee Q \overset{val}{=} Q \vee P$$
$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$
$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0		

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0	0	1

Idempotence and Double Negation

Idempotence

$$P \wedge P \stackrel{val}{=} P$$

$$P \vee P \stackrel{val}{=} P$$

$$P \Rightarrow P \stackrel{val}{\neq} P$$

$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Idempotence and Double Negation

Idempotence

$$P \wedge P \stackrel{val}{=} P$$

$$P \vee P \stackrel{val}{=} P$$

$$P \Rightarrow P \stackrel{val}{\neq} P$$

$$P \Leftrightarrow P \stackrel{val}{\neq} P$$

Double negation

$$\neg\neg P \stackrel{val}{=} P$$

T and F

Inversion

$$\neg T \overset{val}{=} F$$

$$\neg F \overset{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

T and F

Inversion

$$\neg T \overset{val}{=} F$$

$$\neg F \overset{val}{=} T$$

Negation

$$\neg P \overset{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \overset{val}{=} F$$

Excluded Middle

$$P \vee \neg P \overset{val}{=} T$$

T and F

Inversion

$$\neg T \overset{val}{=} F$$

$$\neg F \overset{val}{=} T$$

Negation

$$\neg P \overset{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \overset{val}{=} F$$

T/F - elimination

$$P \wedge T \overset{val}{=}$$

$$P \wedge F \overset{val}{=}$$

$$P \vee T \overset{val}{=}$$

$$P \vee F \overset{val}{=}$$

Excluded Middle

$$P \vee \neg P \overset{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

T/F - elimination

$$P \wedge T \stackrel{val}{=} P$$

$$P \wedge F \stackrel{val}{=} F$$

$$P \vee T \stackrel{val}{=} T$$

$$P \vee F \stackrel{val}{=} P$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \overset{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \overset{val}{=} (P \vee Q) \wedge (P \vee R)$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \stackrel{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \stackrel{val}{=} (P \vee Q) \wedge (P \vee R)$$

De Morgan

$$\neg(P \wedge Q) \stackrel{val}{=} \neg P \vee \neg Q$$

$$\neg(P \vee Q) \stackrel{val}{=} \neg P \wedge \neg Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \overset{val}{=} \neg P \vee Q$$

$$P \vee Q \overset{val}{=} \neg P \Rightarrow Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \overset{val}{=} \neg P \vee Q$$

$$P \vee Q \overset{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \overset{val}{=} \neg Q \Rightarrow \neg P$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \lor Q$$

$$P \lor Q \stackrel{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \stackrel{val}{=} \neg Q \Rightarrow \neg P$$

$$P \Rightarrow Q \stackrel{val}{\neq} \neg P \Rightarrow \neg Q$$

common mistake!

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \overset{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \overset{val}{=}$$

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \overset{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \overset{val}{=} T$$

Calculating with equivalent propositions

(the use of standard equivalences)

Recall...

Definition: Two abstract propositions P and Q are equivalent, notation $P \overset{\text{val}}{=} Q$, iff they induce the same truth-function

on any sequence containing their common variables

Property: The relation $\overset{\text{val}}{=}$ is an equivalence on the set of all abstract propositions.

i.e., for all abstract propositions P, Q, R,
(1) $P \overset{\text{val}}{=} P$; (2) if $P \overset{\text{val}}{=} Q$, then $Q \overset{\text{val}}{=} P$; and
(3) if $P \overset{\text{val}}{=} Q$ and $Q \overset{\text{val}}{=} R$, then $P \overset{\text{val}}{=} R$

Substitution

meta rule

Simple

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]}$$

Sequential

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P][\eta/Q] \stackrel{val}{=} \psi[\xi/P][\eta/Q]}$$

EVERY occurrence of P is substituted!

Simultaneous

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P, \eta/Q] \stackrel{val}{=} \psi[\xi/P, \eta/Q]}$$

The rule of Leibnitz

meta rule

Leibnitz

$$\frac{\phi \overset{val}{=} \psi}{C[\phi] \overset{val}{=} C[\psi]}$$

formula that has ϕ as a sub formula

single occurrence is replaced!

Strengthening and weakening

We had

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{val}{=} Q$, iff

(1) Always when P has truth value 1, also Q has truth value 1, and

(2) Always when Q has truth value 1, also P has truth value 1.

if we relax this, we get strengthening

Strengthening

Definition: The abstract proposition P is stronger than Q, notation $P \overset{val}{\models} Q$, iff
~~(1)~~ Always when P has truth value 1, also Q has truth value 1~~, and~~
~~(2) Always when Q has truth value 1, also P has truth value 1.~~

Q is weaker than P

Strengthening

Definition: The abstract proposition P is stronger than Q, notation $P \models_{val} Q$, iff

always when P has truth value 1,
also Q has truth value 1.

always when P is true,
Q is also true

Q is weaker
than P

Properties

Lemma E1: $P \overset{val}{=} Q$ iff $P \Leftrightarrow Q$ is a tautology.

Lemma EW1: $P \overset{val}{=} Q$ iff $P \overset{val}{\models} Q$ and $Q \overset{val}{\models} P$.

Lemma W2: $P \overset{val}{\models} P$

Lemma W3: If $P \overset{val}{\models} Q$ and $Q \overset{val}{\models} R$ then $P \overset{val}{\models} R$

Lemma W4: $P \overset{val}{\models} Q$ iff $P \Rightarrow Q$ is a tautology.

Standard Weakenings

and-or-weakening

$$P \wedge Q \overset{val}{\models} P$$

$$P \overset{val}{\models} P \vee Q$$

Extremes

$$\mathrm{F} \overset{val}{\models} P$$

$$P \overset{val}{\models} \mathrm{T}$$

Calculating with weakenings
(the use of standard weakenings)

Substitution

just holds

Simple

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P] \overset{val}{\models} \psi[\xi/P]}$$

EVERY occurrence of P is substituted!

Sequential

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P][\eta/Q] \overset{val}{\models} \psi[\xi/P][\eta/Q]}$$

Simultaneous

$$\frac{\phi \overset{val}{\models} \psi}{\phi[\xi/P, \eta/Q] \overset{val}{\models} \psi[\xi/P, \eta/Q]}$$

The rule of Leibnitz

does not hold for weakening!

Leibnitz

$$\cancel{\frac{\phi \overset{val}{\models} \psi}{C[\phi] \overset{val}{\models} C[\psi]}}$$

formula that has ϕ as a sub formula does not hold for weakening!

Leibniz

$$\frac{\phi \overset{val}{\models} \psi}{C[\phi] \overset{val}{\models} C[\psi]}$$

Monotonicity

$$\frac{P \overset{val}{\models} Q}{P \wedge R \overset{val}{\models} Q \wedge R}$$

$$\frac{P \overset{val}{\models} Q}{P \vee R \overset{val}{\models} Q \vee R}$$