

Derivations / Reasoning

Limitations of proofs by calculation

Proofs by calculation are formal and well-structured, but often **undirected** and **not** particularly **intuitive**.

Example

$$\begin{aligned} P \wedge (P \vee Q) &\stackrel{\text{val}}{=} (P \vee F) \wedge (P \vee Q) \\ &\stackrel{\text{val}}{=} P \vee (F \wedge Q) \\ &\stackrel{\text{val}}{=} P \vee F \\ &\stackrel{\text{val}}{=} P \end{aligned}$$

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Conclusions

$$P \wedge (P \vee Q) \stackrel{\text{val}}{=} P \quad P \wedge (P \vee Q) \Leftrightarrow P \stackrel{\text{val}}{=} T$$

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we can prove this more intuitively by reasoning

Conclusions

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An example of a mathematical proof

Theorem

If x^2 is even, then x is even ($x \in \mathbb{Z}$).

Proof

Let $x \in \mathbb{Z}$ be such that x^2 is even.

We need to prove that x is even too.

Assume that x is odd, towards a contradiction.

If x is odd then $x = 2y+1$ for some $y \in \mathbb{Z}$.

Then $x^2 = (2y+1)^2 = 4y^2 + 4y + 1 = 2(2y^2 + 2y) + 1$
and $2y^2 + 2y \in \mathbb{Z}$.

So, x^2 is odd too, and we have a contradiction.

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Thanks to Bas Luttik

Exposing logical structure

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So, x^2 is odd

a contradiction.

So, x is even

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Thanks to Bas Luttik

Single inference rule

Q is a correct conclusion from n premises P_1, \dots, P_n
iff
 $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \stackrel{\text{val}}{\models} Q$

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Note that $T \stackrel{\text{val}}{\models} Q$ means that $Q \stackrel{\text{val}}{=} T$

Q holds
unconditionally

Derivation

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a formal system
based on the single
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for proofs that closely
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Two types of inference rules:

elimination rules

introduction rules

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and one new
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Two types of inference rules:

elimination rules

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for simplifying goals

(particularly useful)
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Conjunction elimination

How do we use a conjunction in a proof?

Conjunction elimination

How do we use a conjunction in a proof?

$$P \wedge Q \stackrel{\text{val}}{=} P$$

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Conjunction elimination

How do we use a conjunction in a proof?

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|| |

(k) $P \wedge Q$

|| |

{ \wedge -elim on (k)}

(m) P

(k < m)

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Implication elimination

How do we use an implication in a proof?

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$$P \Rightarrow Q \stackrel{\text{val}}{\models} ???$$

$$(P \Rightarrow Q) \wedge P \stackrel{\text{val}}{\models} Q$$

Implication elimination

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$$(P \Rightarrow Q) \wedge P \stackrel{\text{val}}{\models} Q$$

|| |

(k) $P \Rightarrow Q$

|| |

(l) P

|| |

{ \Rightarrow -elim on (k) and (l)}

(m) Q

8 (k < m, l < m)

Implication elimination

How do we use an implication in a proof?

\Rightarrow -elimination

|| |
(k) $P \Rightarrow Q$
|| |
(l) P
|| |
{ \Rightarrow -elim on (k) and (l)}
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{ \wedge -intro on (k) and (l)}

(m) $P \wedge Q$

9 (k < m, l < m)

Conjunction introduction

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\wedge -introduction

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(k) P

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(m) $P \wedge Q$

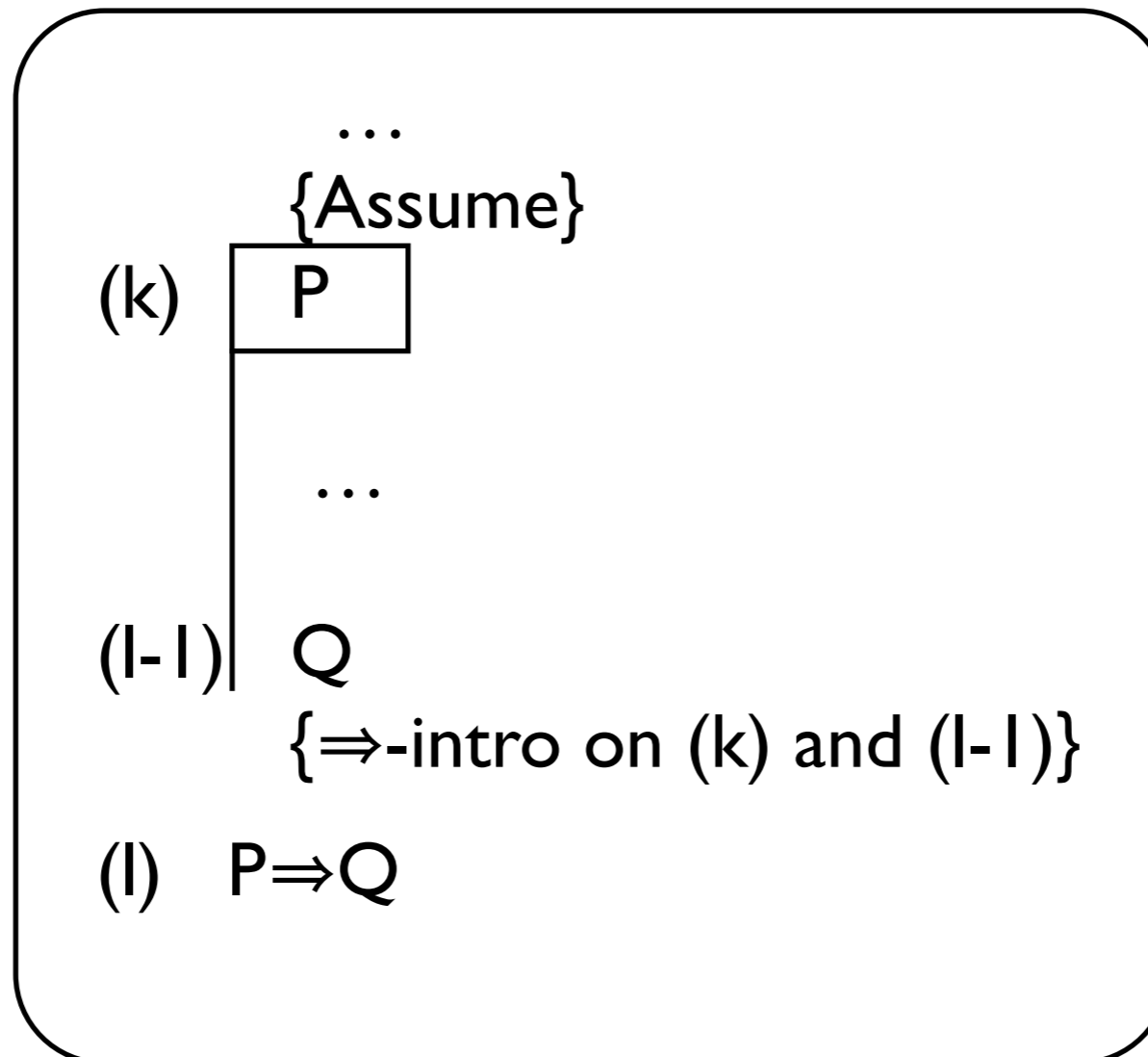
9 (k < m, l < m)

Implication introduction

How do we prove an implication?

Implication introduction

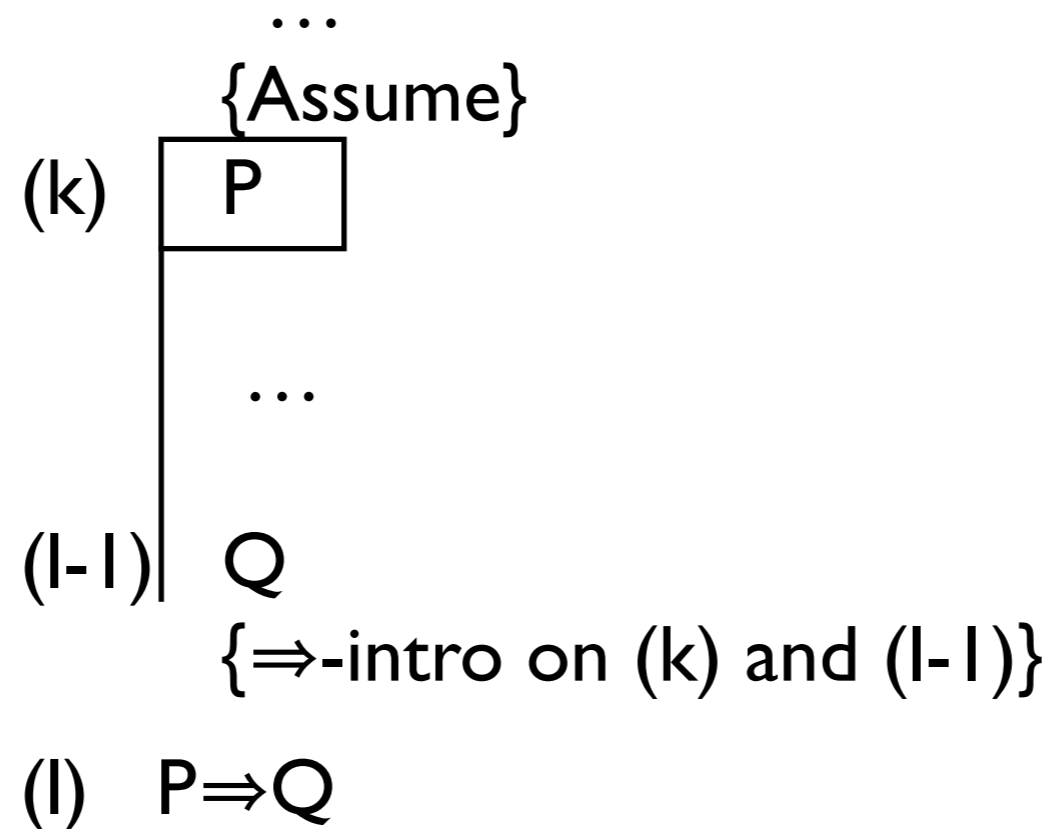
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Implication introduction

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\Rightarrow -introduction



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How do we prove an implication?

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...

{Assume}

(k) P

...

(l-1) Q

{ \Rightarrow -intro on (k) and (l-1)}

(l) $P \Rightarrow Q$

flag shows the validity of a hypothesis

Implication introduction

How do we prove an implication?

truly new
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time for an example!

Negation introduction

How do we prove a negation?

Negation introduction

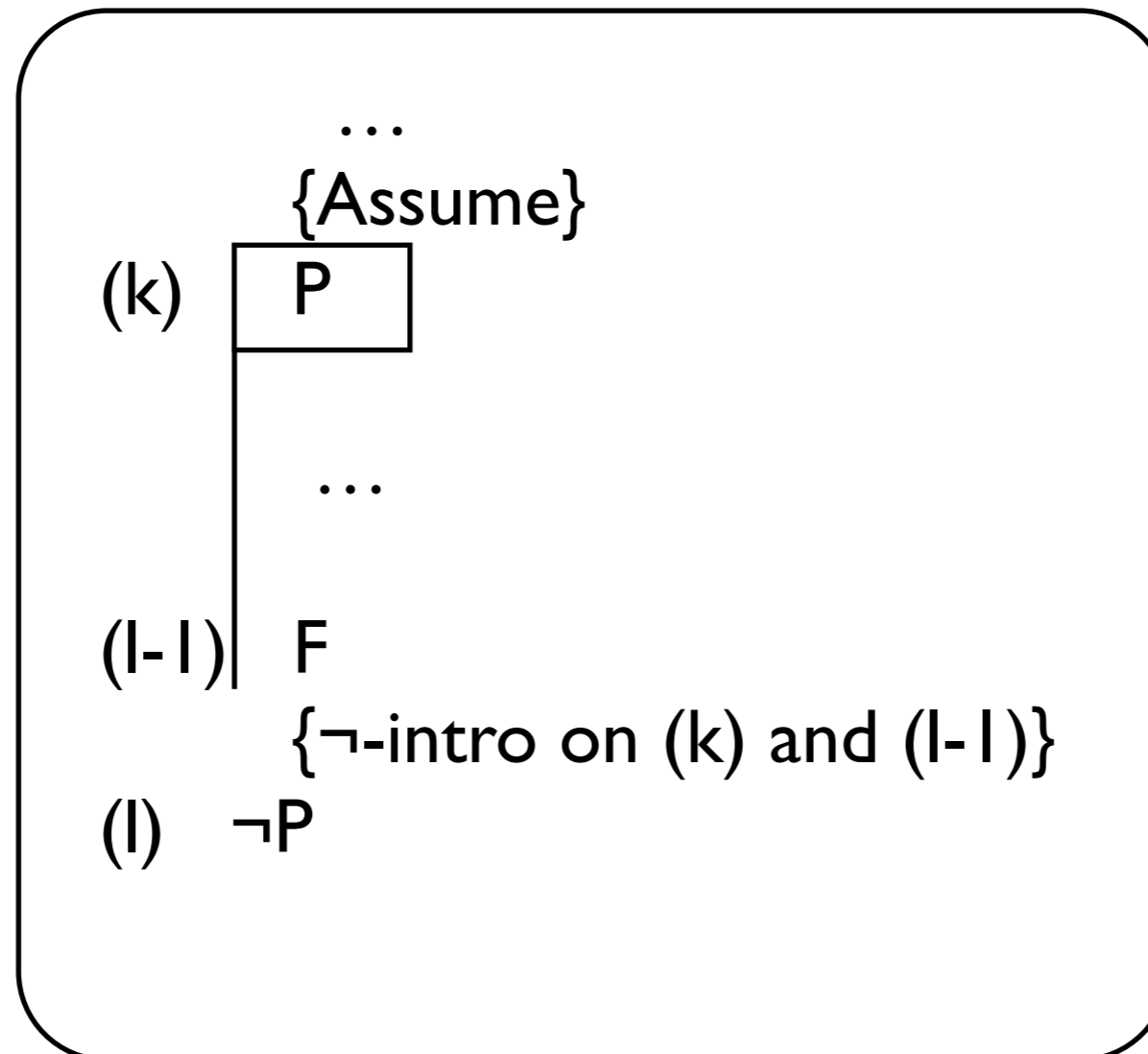
How do we prove a negation?

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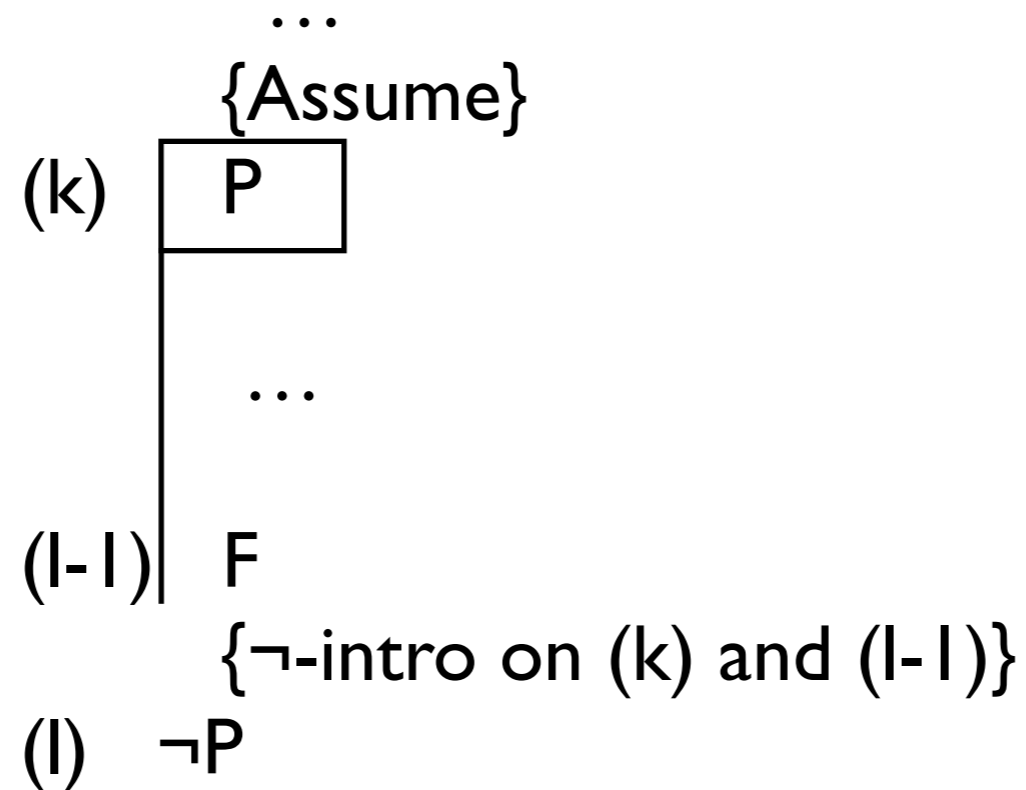


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{Assume}

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(l-1) F

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(l) $\neg P$

$$\neg P \stackrel{\text{val}}{=} P \Rightarrow F$$

\Rightarrow -intro

Negation elimination

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|| |

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{ \neg -elim on (k) and (l)}

(m) F

$_{12} (k < m, l < m)$

Negation elimination

How do we use a negation in a proof?

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\neg -elimination

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\neg -elimination

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(l)	$\neg P$
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(m)	F

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time for an example!

F introduction

How do we prove F?

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...

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...

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(m) F

13 (k < m, l < m)

F introduction

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F-introduction

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F-introduction

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the same as \neg -elim
only intended bottom-up

13 (k < m, l < m)

F elimination

How do we use F in a proof?

F elimination

How do we use F in a proof?

it's very useful!

$$F \stackrel{\text{val}}{=} P$$

F elimination

How do we use F in a proof?

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$$F \stackrel{\text{val}}{\vDash} P$$

|| |
(k) F
|| |
{F-elim on (k)}
(m) P

F elimination

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F-elimination

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(k) F
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14 (k < m)

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Double negation introduction

How do we prove $\neg\neg$?

Double negation introduction

How do we prove $\neg\neg P$?

$$P \stackrel{\text{val}}{=} \neg\neg P$$

Double negation introduction

How do we prove $\neg\neg P$?

$$P \stackrel{\text{val}}{=} \neg\neg P$$

...

(k) P

...

{ $\neg\neg$ -intro on (k)}

(m) $\neg\neg P$

Double negation introduction

How do we prove $\neg\neg P$?

$\neg\neg$ -introduction

...

(k) P

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{ $\neg\neg$ -intro on (k)}

(m) $\neg\neg P$

$$P \stackrel{\text{val}}{=} \neg\neg P$$

Double negation elimination

How do we use $\neg\neg$ in a proof?

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$$\neg\neg P \stackrel{\text{val}}{=} P$$

Double negation elimination

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|| |
(k) $\neg\neg P$
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{ $\neg\neg$ -elim on (k)}
(m) P

Double negation elimination

How do we use $\neg\neg$ in a proof?

$\neg\neg$ -elimination

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(k) $\neg\neg P$
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{ $\neg\neg$ -elim on (k)}
(m) P

$\neg\neg P \stackrel{\text{val}}{=} P$

Proof by contradiction

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If x^2 is even, then x is even ($x \in \mathbb{Z}$).

(sub)goal

Proof

Let $x \in \mathbb{Z}$

generating hypothesis

Assume x^2 is even.

pure hypothesis

Assume that x is odd.

conclusion

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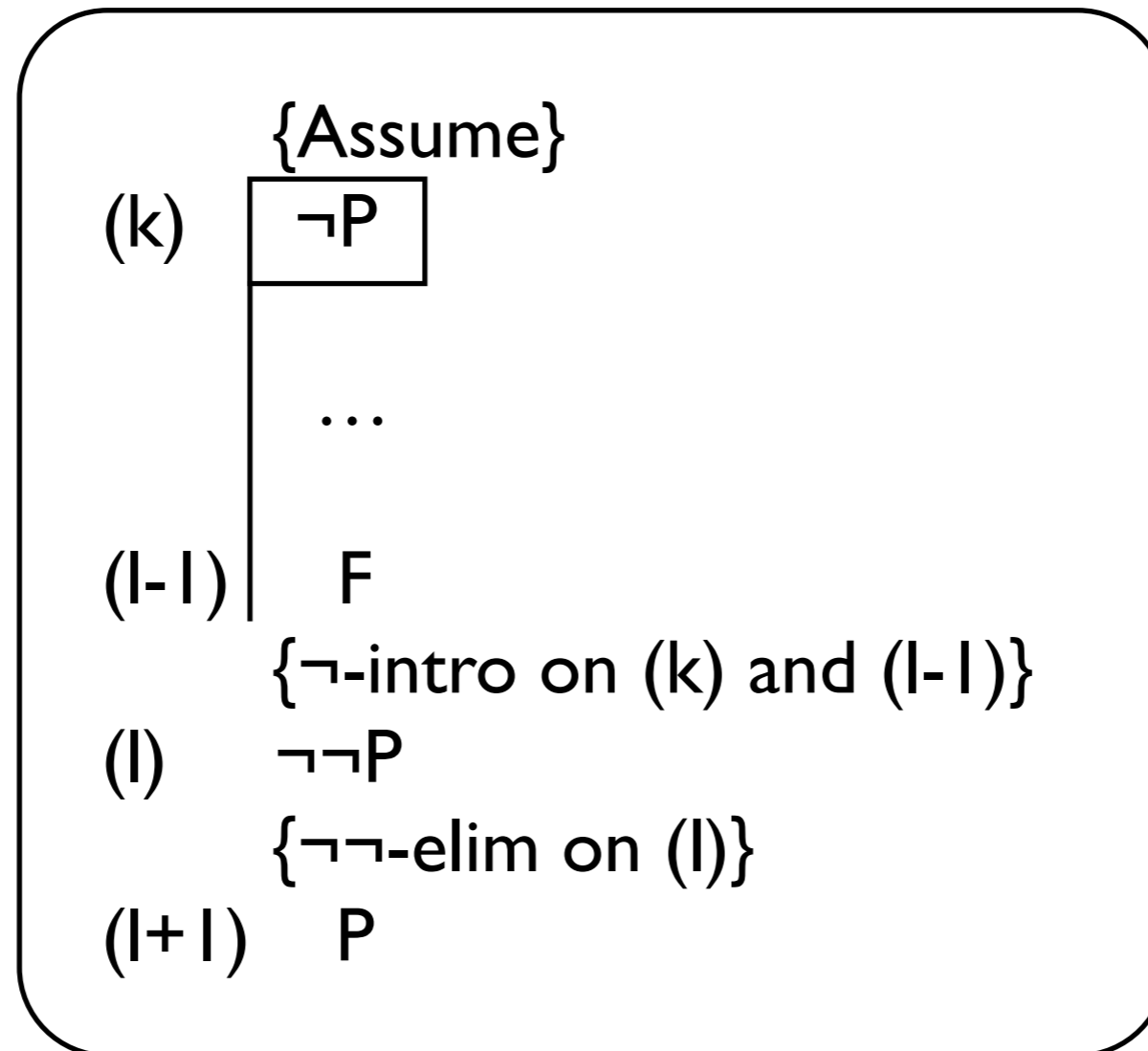
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Proof by contradiction

How do we prove P by a contradiction?

Proof by contradiction

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Proof by contradiction

How do we prove P by a contradiction?

proof by
contradiction

	{Assume}
(k)	$\neg P$
	...
(l-1)	F
	{ \neg -intro on (k) and (l-1)}
(l)	$\neg \neg P$
	{ $\neg \neg$ -elim on (l)}
(l+1)	P

Proof by contradiction

How do we prove P by a contradiction?

proof by contradiction

{Assume}

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(l-1) F
{ \neg -intro on (k) and (l-1)}

(l) $\neg\neg P$
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(l+1) P

$$\neg P \Rightarrow F \stackrel{\text{val}}{=} \neg\neg P \stackrel{\text{val}}{=} P$$

Proof by contradiction

How do we prove P by a contradiction?

proof by contradiction

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(l+1)	P

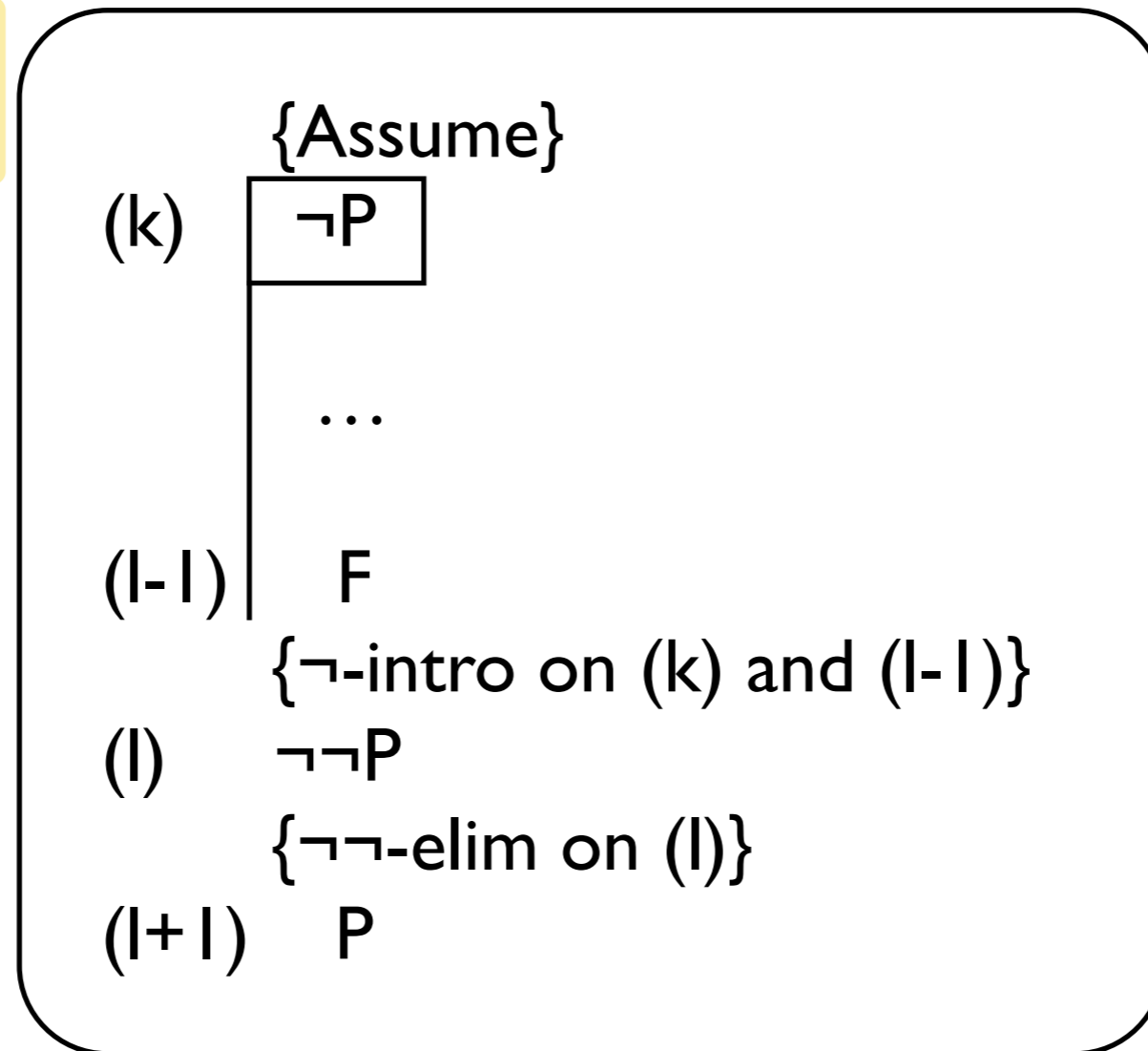
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\neg -intro

Proof by contradiction

How do we prove P by a contradiction?

proof by contradiction



$$\neg P \Rightarrow F \stackrel{\text{val}}{=} \neg\neg P \stackrel{\text{val}}{=} P$$

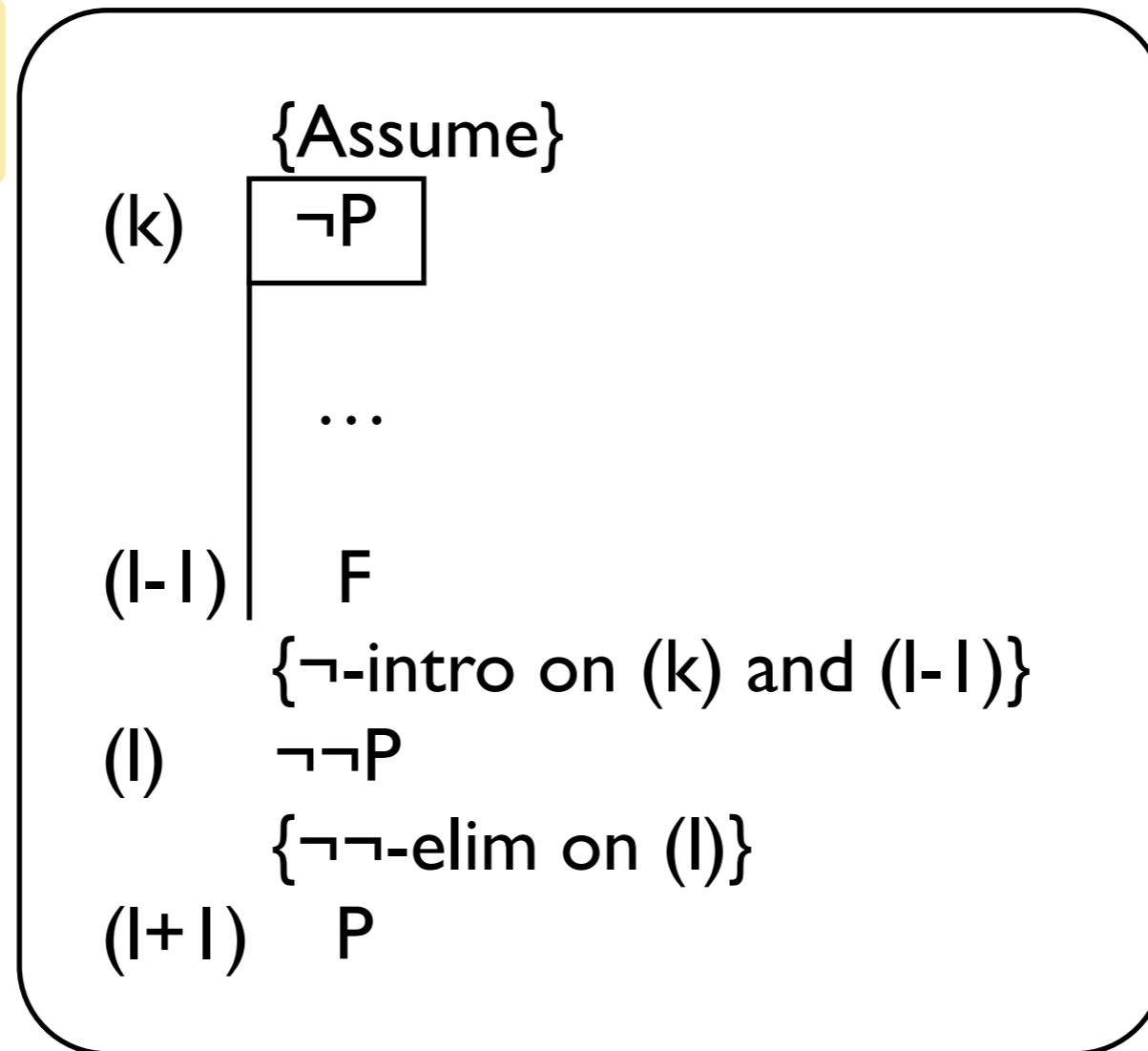
\neg -intro

$\neg\neg$ -elim

Proof by contradiction

How do we prove P by a contradiction?

proof by contradiction



$$\neg P \Rightarrow F \stackrel{\text{val}}{=} \neg\neg P \stackrel{\text{val}}{=} P$$

\neg -intro

$\neg\neg$ -elim

time for an example!

Disjunction introduction

How do we prove a disjunction?

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$$\neg P \Rightarrow Q \stackrel{\text{val}}{\models} P \vee Q$$

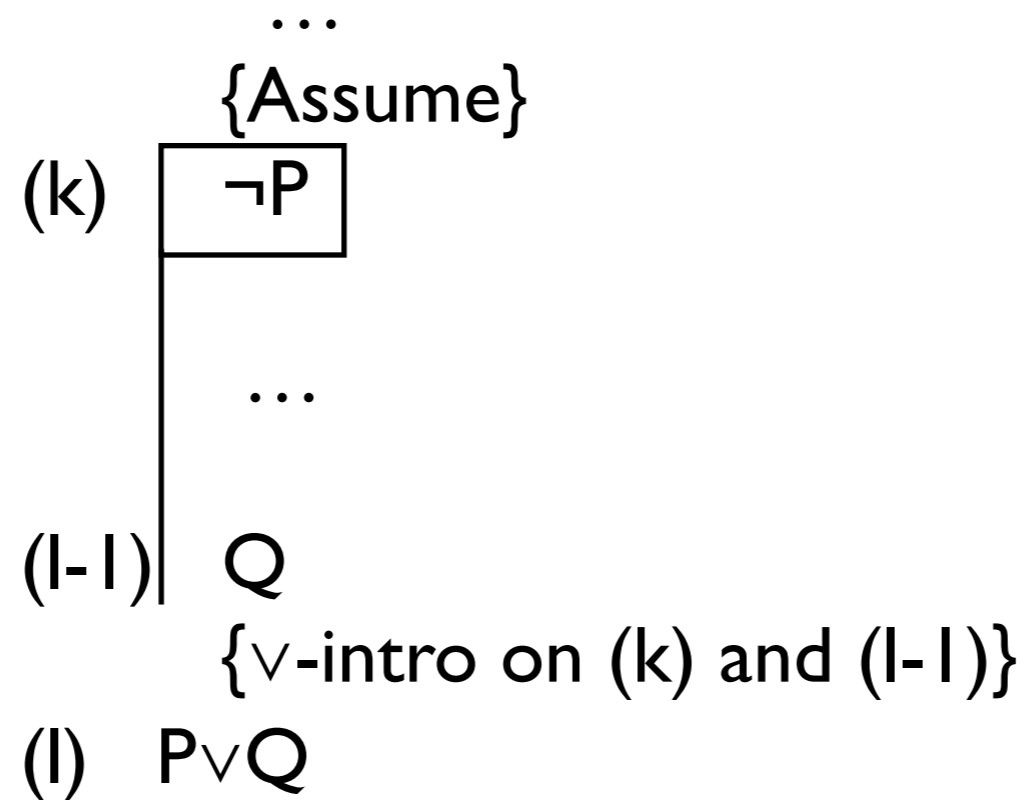
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v-introduction

...

{Assume}

(k) $\neg P$

...

(l-1) Q

{v-intro on (k) and (l-1)}

(l) $P \vee Q$

Disjunction introduction

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(l) $P \vee Q$

Disjunction elimination

How do we use a disjunction in a proof?

Disjunction elimination

How do we use a disjunction in a proof?

$$P \vee Q \stackrel{\text{val}}{\models} \neg P \Rightarrow Q$$

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(k) $P \vee Q$

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{v-elim on (k)}

(m) $\neg P \Rightarrow Q$

Disjunction elimination

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v-elimination

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Proof by case distinction

How do we prove R by a case distinction?

Proof by case distinction

How do we prove R by a case distinction?

|| |

(k) $P \vee Q$

|| |

(l) $P \Rightarrow R$

|| |

(m) $Q \Rightarrow R$

|| |

{case-dist on (k), (l), (m)}

(n) R

Proof by case distinction

How do we prove R by a case distinction?

proof by
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|| |
(k) $P \vee Q$

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(n) {case-dist on (k), (l), (m)}
 R

Proof by case distinction

How do we prove R by a case distinction?

proof by
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$$(P \vee Q) \wedge (P \Rightarrow R) \wedge (Q \Rightarrow R) \stackrel{\text{val}}{\vDash} R$$

|| |
(k) $P \vee Q$

|| |
(l) $P \Rightarrow R$

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Bi-implication introduction

How do we prove a bi-implication?

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P) \stackrel{\text{val}}{\vDash} P \Leftrightarrow Q$$

\Leftrightarrow -introduction

Bi-implication introduction

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{ \Leftrightarrow -intro on (k) and (l)}

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Bi-implication introduction

How do we prove a bi-implication?

$$(P \Rightarrow Q) \wedge (Q \Rightarrow P) \stackrel{\text{val}}{=} P \Leftrightarrow Q$$

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\wedge -intro

Bi-implication elimination

How do we use a bi-implication in a proof?

Bi-implication elimination

How do we use a bi-implication in a proof?

$$P \Leftrightarrow Q \stackrel{\text{val}}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

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\wedge -elim

Derivations / Reasoning with quantifiers

Proving a universal quantification

To prove

$$\forall x [x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$$

Proving a universal quantification

To prove

$$\forall x[x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$$

Proof

Let $x \in \mathbb{Z}$ be arbitrary and assume that $x \geq 2$.

Then, for this particular x , it holds that

$$x^2 - 2x = x(x-2) \geq 0 \quad (\text{Why?})$$

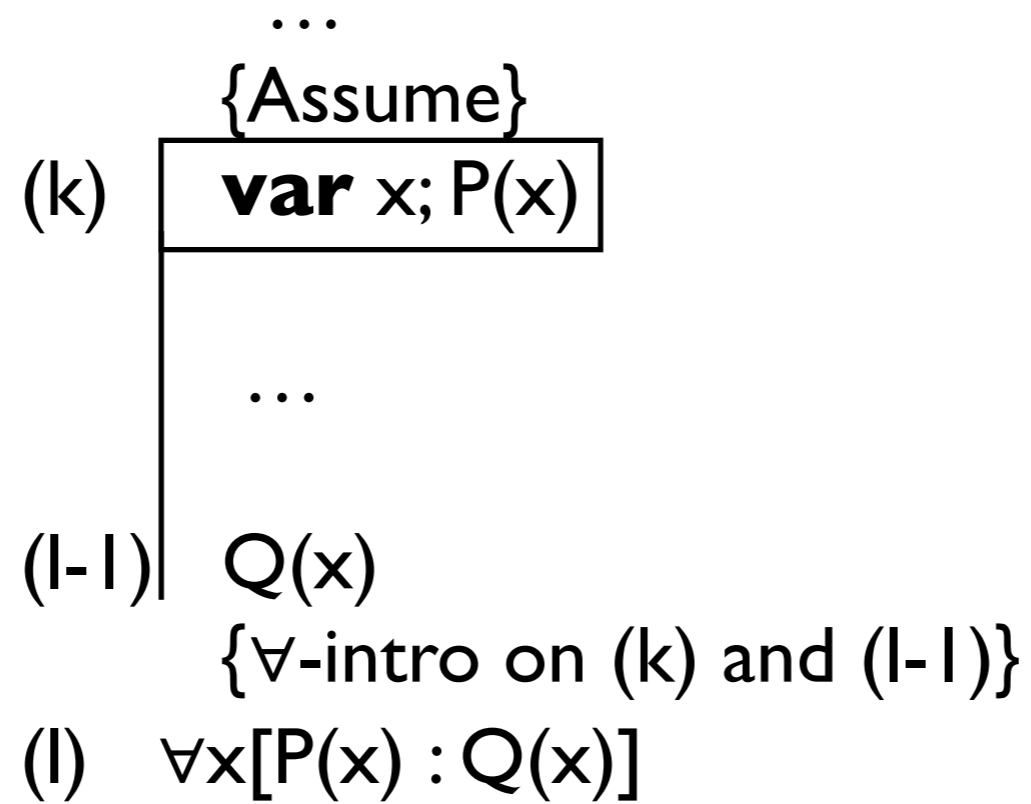
Conclusion: $\forall x[x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$.

\forall introduction

How do we prove a universal quantification?

\forall introduction

How do we prove a universal quantification?



\forall introduction

How do we prove a universal quantification?

\forall -introduction

...

{Assume}

(k) **var** x; P(x)

...

(l-1) Q(x)

{ \forall -intro on (k) and (l-1)}

(l) $\forall x[P(x) : Q(x)]$

\forall introduction

How do we prove a universal quantification?

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flag shows the validity of a hypothesis

\forall introduction

How do we prove a universal quantification?

similar to \Rightarrow -intro
with **generating hypothesis**

\forall -introduction

...

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flag shows the validity of a hypothesis

Using a universal quantification

We know

$$\forall x [x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$$

Using a universal quantification

We know

$$\forall x [x \in \mathbb{Z} \wedge x \geq 2 : x^2 - 2x \geq 0]$$

Whenever we encounter an $a \in \mathbb{Z}$ such that $a \geq 2$,
we can conclude that $a^2 - 2a \geq 0$.

For example, $(52387^2 - 2 \cdot 52387) \geq 0$
since $52387 \in \mathbb{Z}$ and $52387 \geq 2$.

\forall elimination

How do we use a universal quantification in a proof?

\forall elimination

How do we use a universal quantification in a proof?

similar to
implication
but we need
a witness

\forall elimination

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(k) $\forall x[P(x) : Q(x)]$
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{ \forall -elim on (k) and (l)}
(m) $Q(a)$

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a is an object (variable, number,..) which is "known" in line (l)

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time for an example!

\exists introduction

How do we prove an existential quantification?

\exists introduction

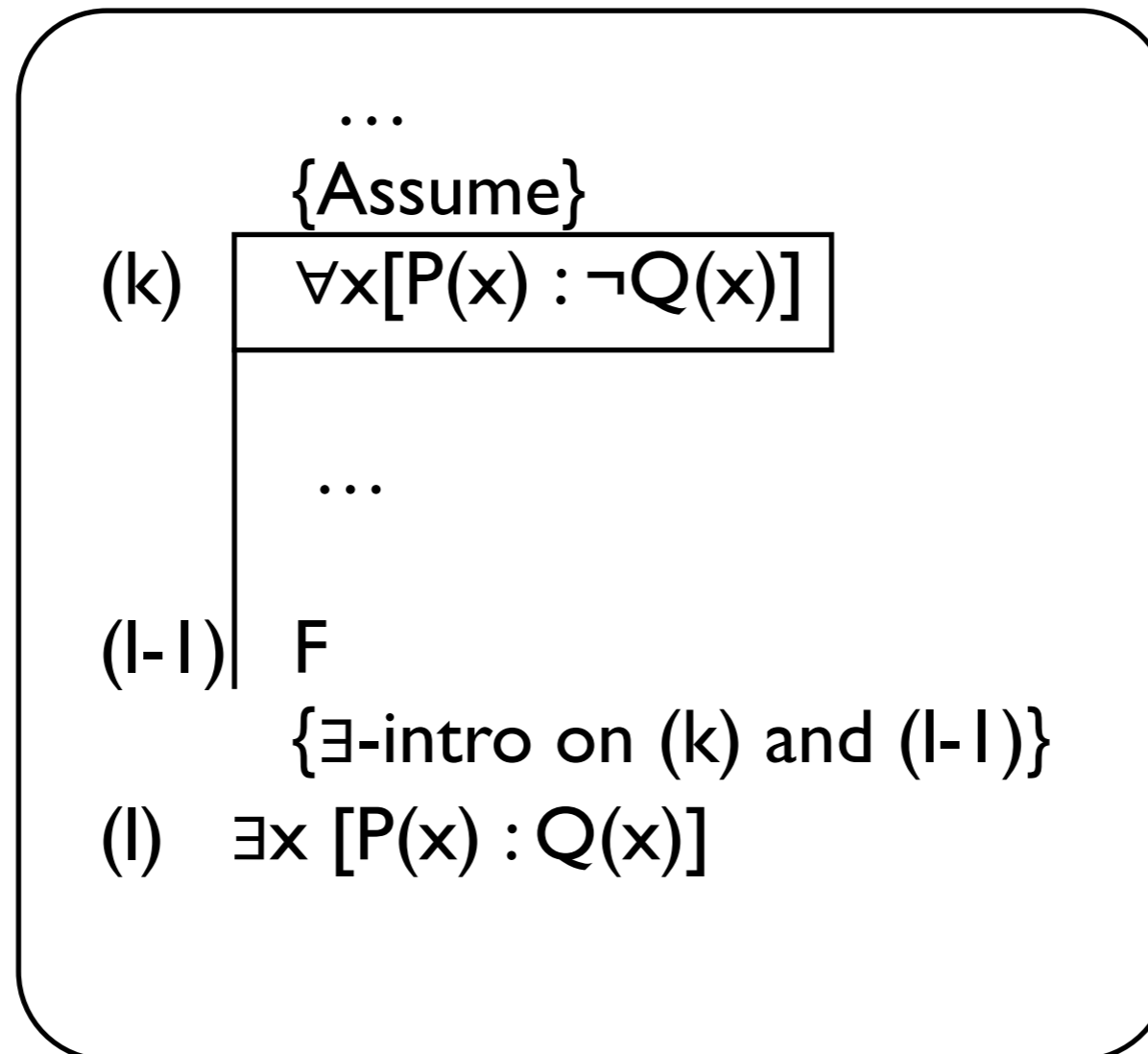
How do we prove an existential quantification?

$$\neg \forall x [P(x) : \neg Q(x)] \stackrel{\text{val}}{\equiv} \exists x [P(x) : Q(x)]$$

∃ introduction

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\exists introduction

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\exists -introduction

...

{Assume}

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...

(l-1) F

{ \exists -intro on (k) and (l-1)}

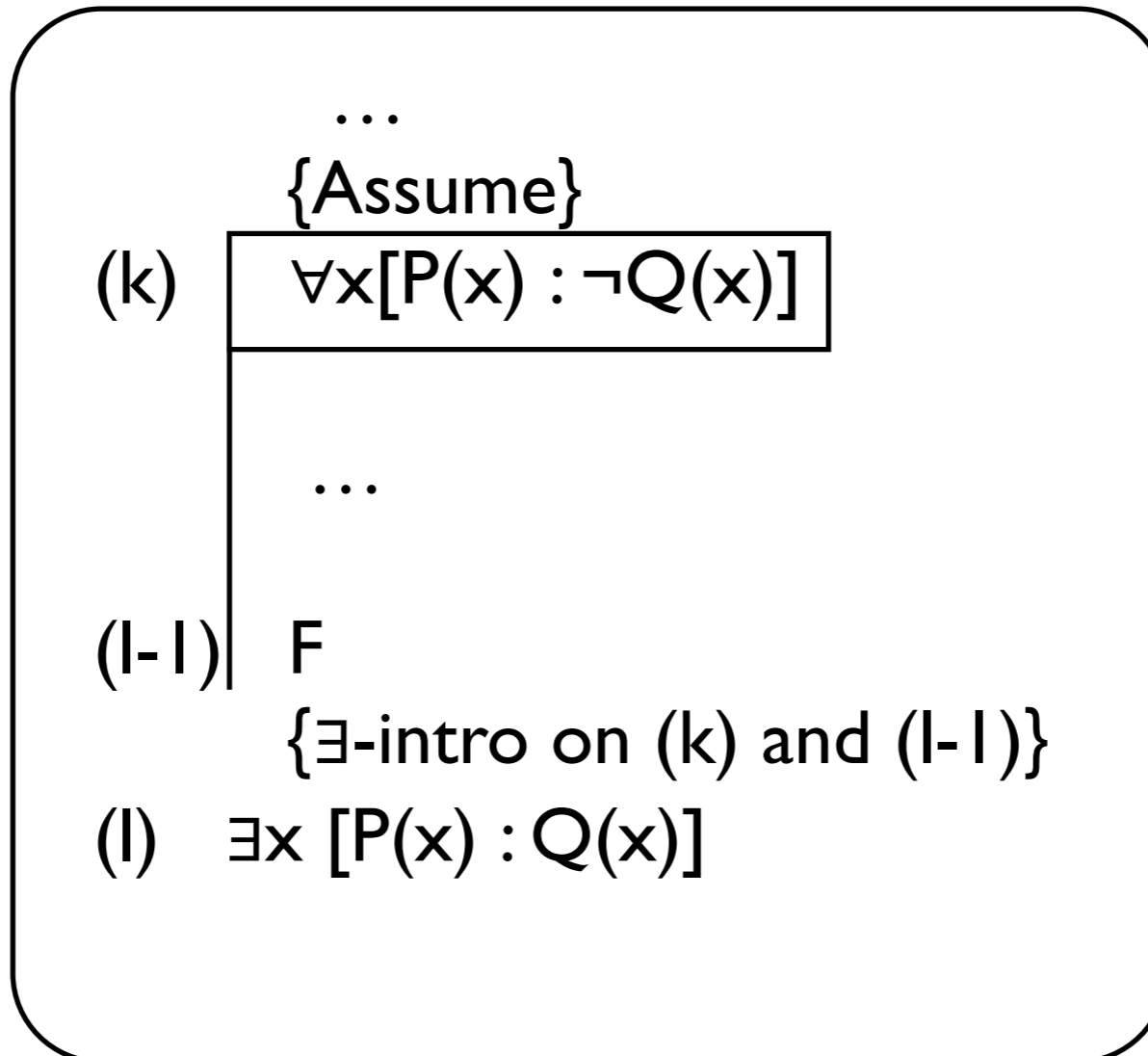
(l) $\exists x [P(x) : Q(x)]$

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How do we prove an existential quantification?

$$\neg \forall x [P(x) : \neg Q(x)] \stackrel{\text{val}}{\equiv} \exists x [P(x) : Q(x)]$$

\exists -introduction



and \neg -intro

\exists elimination

How do we use an existential quantification in a proof?

\exists elimination

How do we use an existential quantification in a proof?

$$\exists x [P(x) : Q(x)] \stackrel{val}{=} \neg \forall x [P(x) : \neg Q(x)]$$

\exists elimination

How do we use an existential quantification in a proof?

$$\begin{aligned} \exists x [P(x) : Q(x)] &\stackrel{\text{val}}{=} \\ \neg \forall x [P(x) : \neg Q(x)] & \end{aligned}$$

and \neg -
elimination

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time for an
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Proofs with \exists -introduction and \exists -elimination are unnecessarily long and cumbersome...

Proofs with \exists -introduction and \exists -elimination are unnecessarily long and cumbersome...



There are alternatives!

Proving an existential quantification

To prove

$$\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$$

Proving an existential quantification

To prove

$$\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$$

Proof

It suffices to find a witness, i.e., an $x \in \mathbb{Z}$ satisfying

$$x^3 - 2x - 8 \geq 0.$$

$x = 3$ is a witness, since $3 \in \mathbb{Z}$ and $3^3 - 2 \cdot 3 - 8 = 13 \geq 0$

Conclusion: $\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$.

Proving an existential quantification

To prove

$$\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$$

Proof

It suffices to find a witness, i.e., an $x \in \mathbb{Z}$ satisfying
 $x^3 - 2x - 8 \geq 0$.

$x = 3$ is a witness, since $3 \in \mathbb{Z}$ and $3^3 - 2 \cdot 3 - 8 = 13 \geq 0$

Conclusion: $\exists x[x \in \mathbb{Z} : x^3 - 2x - 8 \geq 0]$.

also $x = 5$ is a witness...

Alternative \exists introduction

How do we prove an existential quantification?

Alternative \exists introduction

How do we prove an existential quantification?

by finding
a witness

Alternative \exists introduction

How do we prove an existential quantification?

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(k) $P(a)$

...

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{ \exists^* -intro on (k) and (l)}

(m) $\exists x [P(x) : Q(x)]$

Alternative \exists introduction

How do we prove an existential quantification?

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\exists^* -introduction

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Alternative \exists introduction

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strategy: wait until a witness
object appears

Alternative \exists introduction

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\exists^* -introduction

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(m) $\exists x [P(x) : Q(x)]$

strategy: wait until a witness
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does not
always work

Using an existential quantification

We know

$$\exists x[x \in \mathbb{R} : a - x < 0 < b - x]$$

Using an existential quantification

We know

$$\exists x[x \in \mathbb{R} : a - x < 0 < b - x]$$

We can declare an $x \in \mathbb{Z}$ (a witness) such that

$$a - x < 0 < b - x$$

and use it further in the proof. For example:

From $a - x < 0$, we get $a < x$.

From $b - x > 0$, we get $x < b$.

Hence, $a < b$.

Alternative \exists elimination

How do we use an existential quantification in a proof?

Alternative \exists elimination

How do we use an existential quantification in a proof?

we pick a witness

Alternative \exists elimination

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{ \exists^* -elim on (k)}

(m) Pick x with $P(x)$ and $Q(x)$

Alternative \exists elimination

How do we use an existential quantification in a proof?

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\exists^* -elimination

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x must be new!

Alternative \exists elimination

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