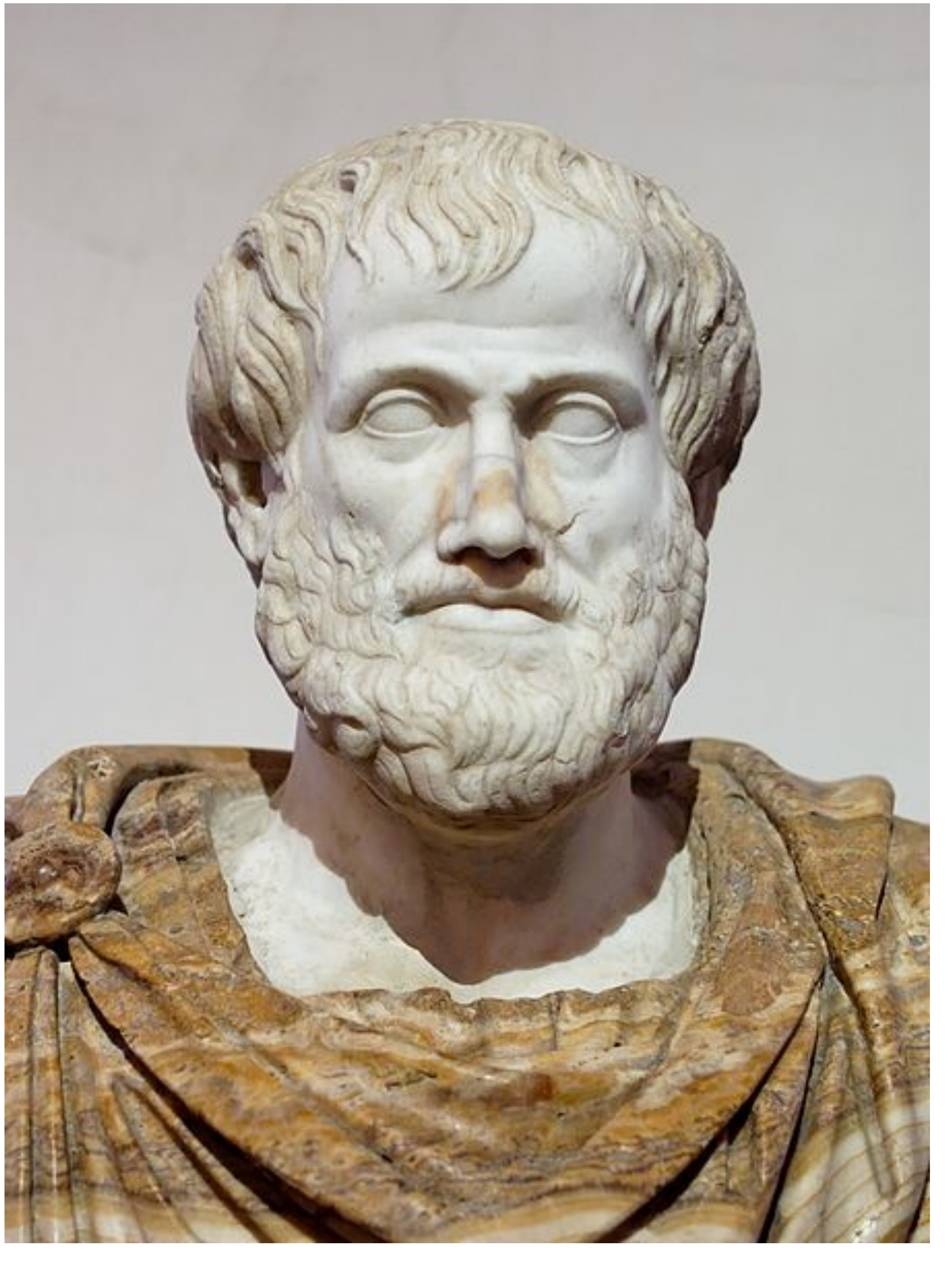
Logic

In the beginning

Aristotle +/- 350 B.C.

Organon

19 syllogisms



Logic = study of correct reasoning

In the beginning

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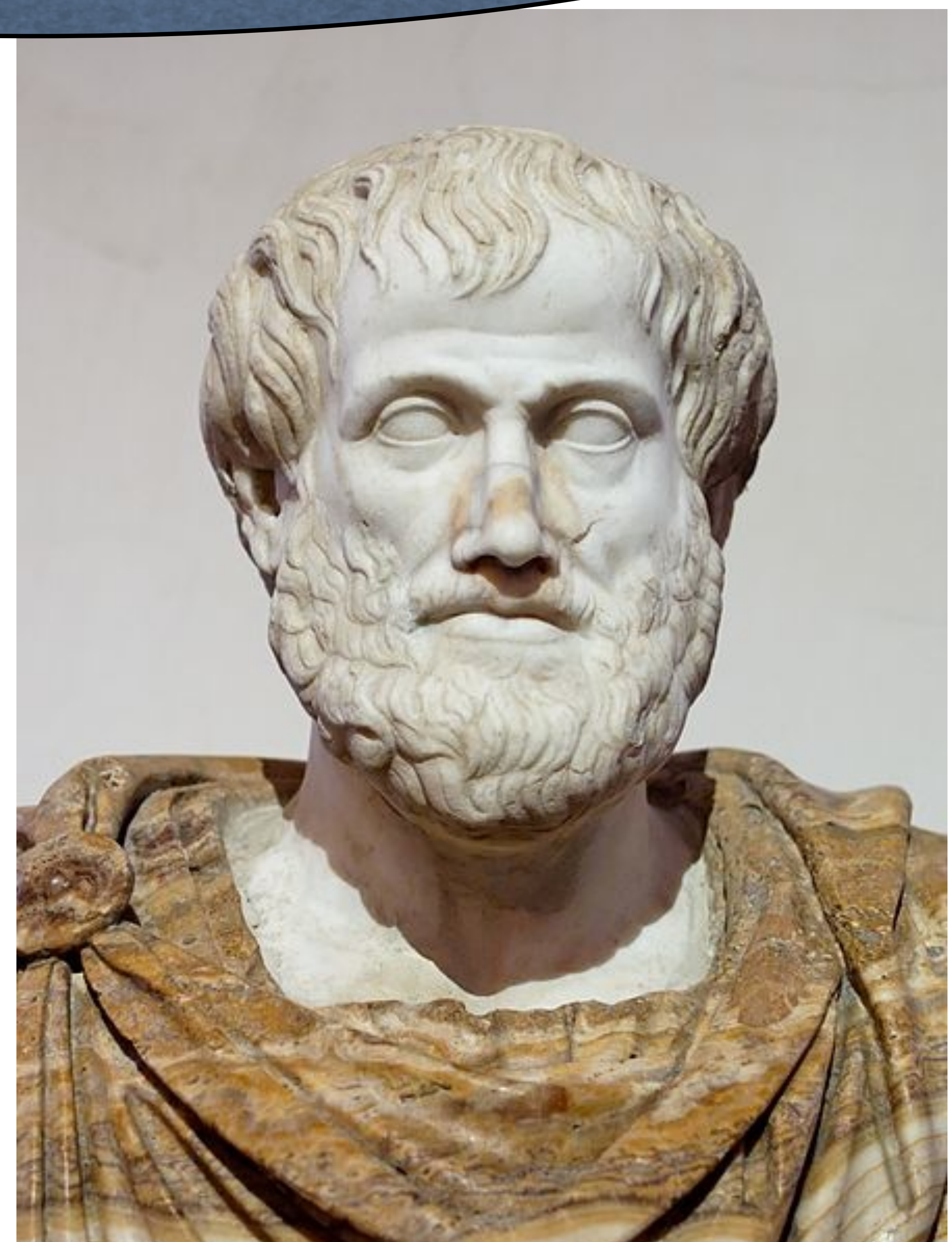

Barbara syllogism

All K's are L's
All L's are M's

All K's are M's

Barbara syllogism

only later called so, in the Middle Ages

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from the two
premises

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from the two premises

one can **always** conclude the conclusion

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from the two premises

one can **always** conclude the conclusion

independent of what the parameters K,L,M are

Barbara syllogism

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All K's are L's
All L's are M's

All K's are M's

from the two premises

one can always conclude the conclusion

independent of what the parameters K,L,M are

Logic (Logos, Greek for word, understanding, reason) deals with general reasoning laws in the form of formulas with parameters.

Propositions

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Def. A proposition (Aussage) is a grammatically correct sentence that is either true or false.

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logic deals with patterns!
what matters are not particular propositions but the way in which (abstract) propositions are combined and what follows from them

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logic deals with patterns! what matters are not particular propositions but the way in which (abstract) propositions are combined and what follows from them

Connectives

- $\wedge$ for "and"
- $\vee$ for "or"
- $\neg$ for "not"
- $\Rightarrow$ for "if .. then" or "implies"
- $\Leftrightarrow$ for "if and only if"

Abstract propositions

Abstract propositions

Definition

Basis	Propositional variables are abstract propositions.
Step (Case 1)	If P is an abstract proposition, then so is $(\neg P)$.
Step (Case 2)	If P and Q are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

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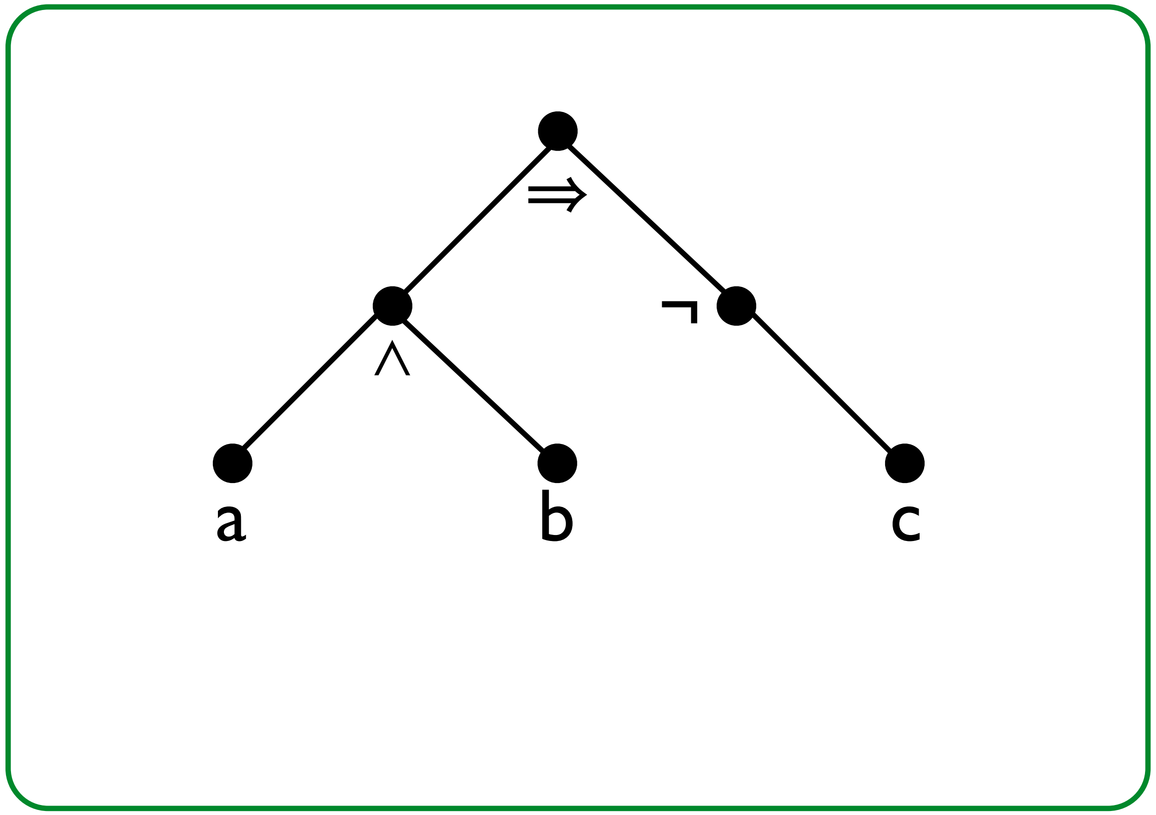
a recursive/inductive definition

…and their structure



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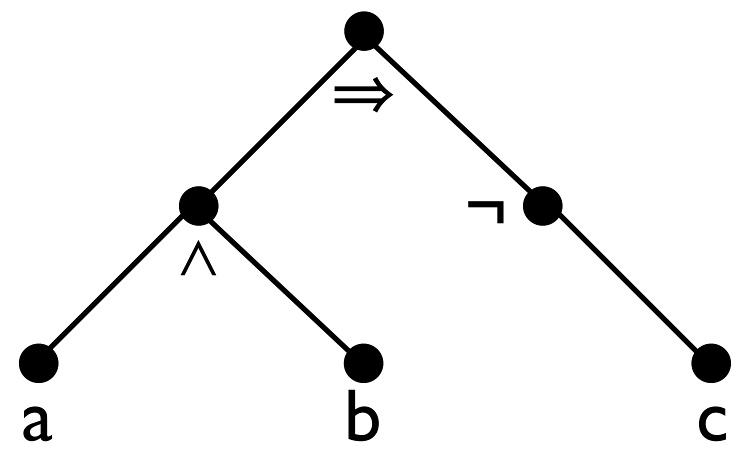
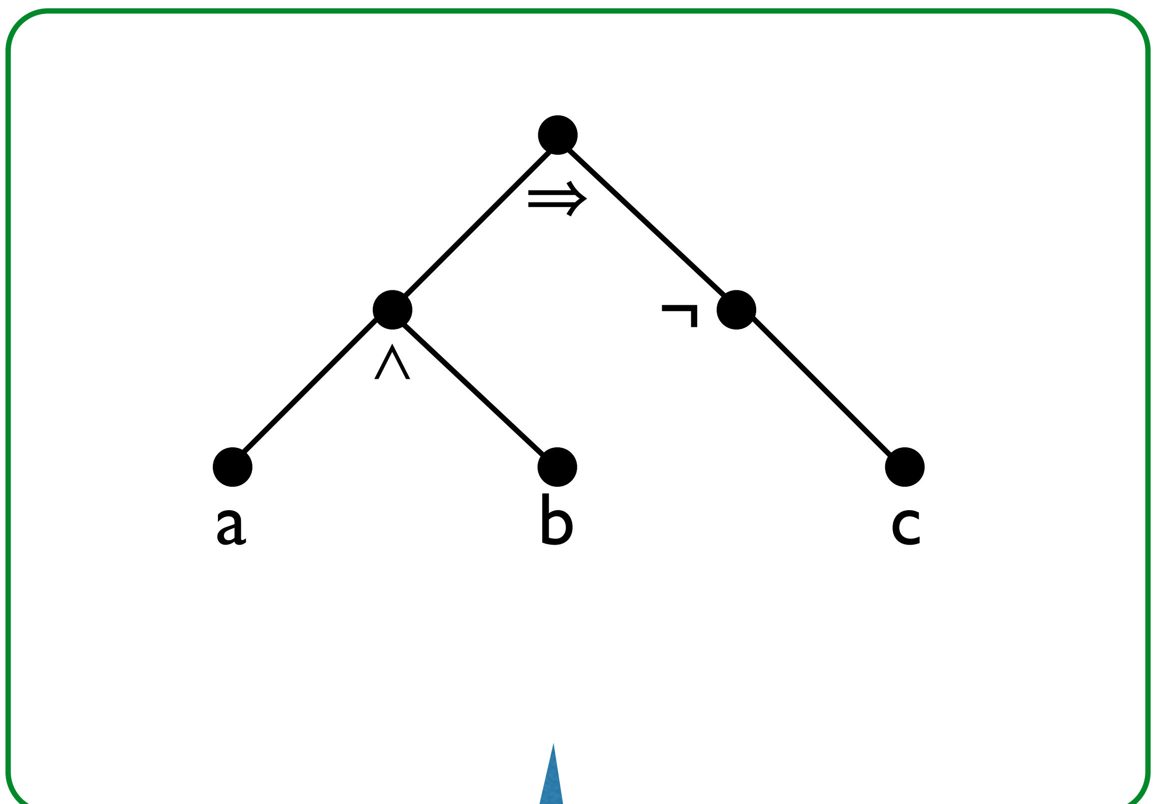


the tree of
$((a \wedge b) \Rightarrow (\neg c))$

…and their structure



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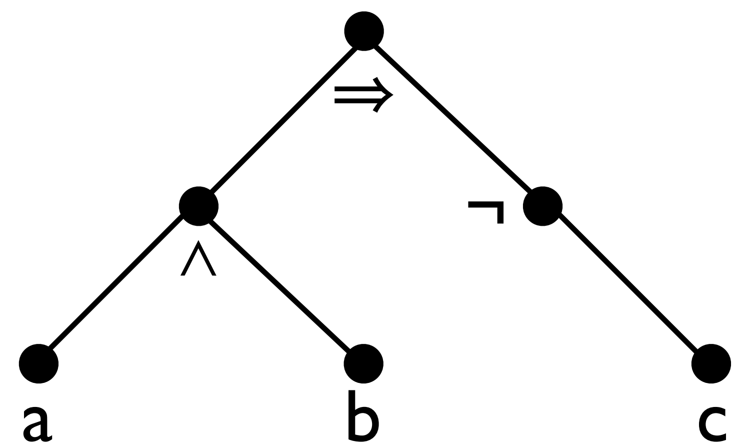

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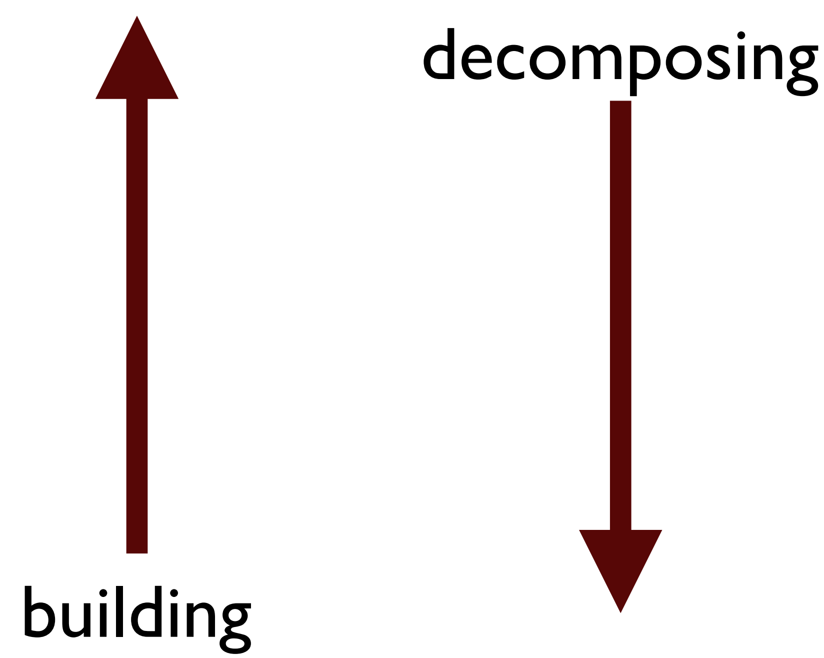
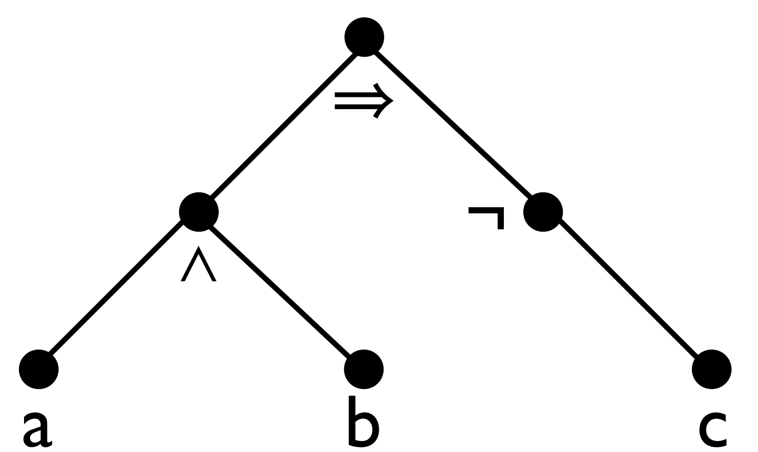


…and their structure



decomposing

building

the tree of
$((a \wedge b) \Rightarrow (\neg c))$

tree representation
(no need of
parenthesis)

Dropping parenthesis

$\neg$
$\wedge \quad \vee$
$\Rightarrow$
$\Leftrightarrow$

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$\neg$
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$\Rightarrow$
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priority schema
(top binds the most)

Dropping parenthesis

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priority schema
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increasing priority

Dropping parenthesis

$\neg$

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$\Leftrightarrow$

priority schema
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increasing priority

decreasing priority

Dropping parenthesis

$\neg$
$\wedge \quad \vee$
$\Rightarrow$
$\Leftrightarrow$

priority schema
(top binds the most)

decreasing priority

increasing priority

Example: $((a \wedge b) \Rightarrow (\neg c))$

becomes

$a \wedge b \Rightarrow \neg c$

Truth tables

Conjunction

P	Q	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

Truth tables

Conjunction

P	Q	P∧Q
0	0	0
0	1	0
1	0	0
1	1	1

Truth tables

Conjunction

P	Q	$P \wedge Q$
0	0	0
0	1	0
1	0	0
1	1	1

only true when both P and Q are true

Truth tables

Disjunction

P	Q	P∨Q
0	0	0
0	1	1
1	0	1
1	1	1

Truth tables

Disjunction

P	Q	P∨Q
0	0	0
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Truth tables

Disjunction

P	Q	P∨Q
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true when either P or Q or both are true

Truth tables

Negation

Truth tables

Negation

unary connective

Truth tables

Negation

unary connective

P	¬P
0	1
1	0

Truth tables

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unary connective

P	¬P
0	1
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Truth tables

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unary connective

P	¬P
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1	0

true when P is false

Truth tables

Implication

Truth tables

Implication

needs more attention

Truth tables

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needs more attention

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

Truth tables

Implication

needs more attention

P	Q	$P \Rightarrow Q$
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0	1	1
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Truth tables

Implication

needs more attention

P	Q	$P \Rightarrow Q$
0	0	1
0	1	1
1	0	0
1	1	1

only false when P is true and Q is false

Truth tables

Bi-implication

Truth tables

Bi-implication

$P \Leftrightarrow Q$

is $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

Truth tables

Bi-implication

$P \Leftrightarrow Q$

is $(P \Rightarrow Q) \wedge (Q \Rightarrow P)$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
0	0	1	1	1
0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

Truth tables

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0	0	1	1	1
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1	1	1	1	1

Truth tables

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P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$	$P \Leftrightarrow Q$
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0	1	1	0	0
1	0	0	1	0
1	1	1	1	1

true when P and Q have the same truth value

Truth-functions

Def. A truth-function or Boolean function is a function

$$f: \{0,1\}^n \longrightarrow \{0,1\}$$

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Notation in the book…

$$\left\{\begin{array}{l} \quad a, b \\ (0,0) \longmapsto 0 \\ (0,1) \longmapsto 1 \\ (1,0) \longmapsto 0 \\ (1,1) \longmapsto 1 \end{array}\right.$$

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$P(a,b)$: $(a \wedge b) \vee b$

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$a_1, .. a_n$ are the variables in P (and more) ordered in a sequence

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Truth-functions

$a_1, .. a_n$ are the variables in P (and more) ordered in a sequence

Property: Every abstract proposition $P(a_1,..,a_n)$ with ordered and specified variables induces a truth-function.

Note:

The sequence of specified variables matters!

$P(a,b,c)$: $(a \wedge b) \vee b$

induces

$$
\left\{
\begin{array}{l}
\quad a, b, c \\
(0,0,0) \longmapsto 0 \\
(0,0,1) \longmapsto 0 \\
(0,1,0) \longmapsto 1 \\
(0,1,1) \longmapsto 1 \\
(1,0,0) \longmapsto 0 \\
(1,0,1) \longmapsto 0 \\
(1,1,0) \longmapsto 1 \\
(1,1,1) \longmapsto 1
\end{array}
\right.
$$

Equivalence of propositions

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Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{val}{=} Q$, iff they induce the same truth-function

on any sequence containing their common variables

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i.e., for all abstract propositions P, Q, R,
(1) $P \stackrel{val}{=} P$; (2) if $P \stackrel{val}{=} Q$, then $Q \stackrel{val}{=} P$; and
(3) if $P \stackrel{val}{=} Q$ and $Q \stackrel{val}{=} R$, then $P \stackrel{val}{=} R$

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
0	0				
0	1				
1	0				
1	1				

Example

Are the following equivalent? $b \wedge \neg b$ and $c \wedge \neg c$

b	c	$\neg b$	$\neg c$	$b \wedge \neg b$	$c \wedge \neg c$
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0	1	1			
1	0	0			
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0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

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0	1	1	0	0	0
1	0	0	1	0	0
1	1	0	0	0	0

Their truth values are the same, so they are equivalent

$$b \wedge \neg b \stackrel{val}{=} c \wedge \neg c$$

Tautologies and contradictions

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Def. An abstract proposition P is a tautology iff its truth-function is constant 1.

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Def. An abstract proposition P is a contingency iff it is neither a tautology nor a contradiction.

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all contradictions are equivalent

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Tautologies and contradictions

Def. An abstract proposition P is a tautology iff its truth-function is constant 1.

all tautologies are equivalent

Def. An abstract proposition P is a contradiction iff its truth-function is constant 0.

all contradictions are equivalent

but not all contingencies!

Def. An abstract proposition P is a contingency iff it is neither a tautology nor a contradiction.

Abstract propositions

Definition

Basis (Case 1) T and F are abstract propositions.
Basis (Case 2) Propositional variables are abstract propositions.

Step (Case 1) If P is an abstract proposition, then so is $(\neg P)$.
Step (Case 2) If P and Q are abstract propositions, then so are $(P \wedge Q)$, $(P \vee Q)$, $(P \Rightarrow Q)$, and $(P \Leftrightarrow Q)$.

a recursive/inductive definition

Propositional Logic
Standard Equivalences

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

$$P \Rightarrow Q \overset{val}{\neq} Q \Rightarrow P$$

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
0	1	1	0

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

Commutativity and Associativity

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$$(P \wedge Q) \wedge R \stackrel{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \stackrel{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \stackrel{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \stackrel{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$
$$P \vee Q \overset{val}{=} Q \vee P$$
$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$
$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$
$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$

Commutativity and Associativity

Commutativity

$$P \wedge Q \overset{val}{=} Q \wedge P$$

$$P \vee Q \overset{val}{=} Q \vee P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \wedge Q) \wedge R \overset{val}{=} P \wedge (Q \wedge R)$$

$$(P \vee Q) \vee R \overset{val}{=} P \vee (Q \vee R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0		

Commutativity and Associativity

Commutativity

$$P \land Q \overset{val}{=} Q \land P$$

$$P \lor Q \overset{val}{=} Q \lor P$$

$$P \Leftrightarrow Q \overset{val}{=} Q \Leftrightarrow P$$

Associativity

$$(P \land Q) \land R \overset{val}{=} P \land (Q \land R)$$

$$(P \lor Q) \lor R \overset{val}{=} P \lor (Q \lor R)$$

$$(P \Leftrightarrow Q) \Leftrightarrow R \overset{val}{=} P \Leftrightarrow (Q \Leftrightarrow R)$$

$$(P \Rightarrow Q) \Rightarrow R \overset{val}{\neq} P \Rightarrow (Q \Rightarrow R)$$

P	Q	R	$(P \Rightarrow Q) \Rightarrow R$	$P \Rightarrow (Q \Rightarrow R)$
0	1	0	0	1

Idempotence and Double Negation

Idempotence

$$P \wedge P \overset{val}{=} P$$

$$P \vee P \overset{val}{=} P$$

$$P \Rightarrow P \overset{val}{\neq} P$$

$$P \Leftrightarrow P \overset{val}{\neq} P$$

Idempotence and Double Negation

Idempotence

$$P \wedge P \overset{val}{=} P$$

$$P \vee P \overset{val}{=} P$$

$$P \Rightarrow P \overset{val}{\neq} P$$

$$P \Leftrightarrow P \overset{val}{\neq} P$$

Double negation

$$\neg\neg P \overset{val}{=} P$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$

$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

T and F

Inversion

$$\neg T \stackrel{val}{=} F$$
$$\neg F \stackrel{val}{=} T$$

Negation

$$\neg P \stackrel{val}{=} P \Rightarrow F$$

Contradiction

$$P \wedge \neg P \stackrel{val}{=} F$$

T/F - elimination

$$P \wedge T \stackrel{val}{=}$$
$$P \wedge F \stackrel{val}{=}$$
$$P \vee T \stackrel{val}{=}$$
$$P \vee F \stackrel{val}{=}$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

T and F

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$$\neg F \stackrel{val}{=} T$$

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T/F - elimination

$$P \wedge T \stackrel{val}{=} P$$
$$P \wedge F \stackrel{val}{=} F$$
$$P \vee T \stackrel{val}{=} T$$
$$P \vee F \stackrel{val}{=} P$$

Excluded Middle

$$P \vee \neg P \stackrel{val}{=} T$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \stackrel{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \stackrel{val}{=} (P \vee Q) \wedge (P \vee R)$$

Distributivity, De Morgan

Distributivity

$$P \wedge (Q \vee R) \stackrel{val}{=} (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) \stackrel{val}{=} (P \vee Q) \wedge (P \vee R)$$

De Morgan

$$\neg(P \wedge Q) \stackrel{val}{=} \neg P \vee \neg Q$$

$$\neg(P \vee Q) \stackrel{val}{=} \neg P \wedge \neg Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \stackrel{val}{=} \neg P \vee Q$$

$$P \vee Q \stackrel{val}{=} \neg P \Rightarrow Q$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \overset{val}{=} \neg P \vee Q$$

$$P \vee Q \overset{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \overset{val}{=} \neg Q \Rightarrow \neg P$$

Implication and Contraposition

Implication

$$P \Rightarrow Q \overset{val}{=} \neg P \vee Q$$

$$P \vee Q \overset{val}{=} \neg P \Rightarrow Q$$

Contraposition

$$P \Rightarrow Q \overset{val}{=} \neg Q \Rightarrow \neg P$$

$$P \Rightarrow Q \overset{val}{\neq} \neg P \Rightarrow \neg Q$$

common mistake!

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \overset{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Bi-implication and Self-equivalence

Bi-implication

$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \stackrel{val}{=}$$

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$$P \Leftrightarrow Q \stackrel{val}{=} (P \Rightarrow Q) \wedge (Q \Rightarrow P)$$

Self-equivalence

$$P \Leftrightarrow P \stackrel{val}{=} T$$

Calculating with equivalent propositions

(the use of standard equivalences)

Recall…

Definition: Two abstract propositions P and Q are equivalent, notation $P \overset{\text{val}}{=} Q$, iff they induce the same truth-function

on any sequence containing their common variables

Property: The relation $\overset{\text{val}}{=}$ is an equivalence on the set of all abstract propositions.

i.e., for all abstract propositions P, Q, R,
(1) $P \overset{\text{val}}{=} P$; (2) if $P \overset{\text{val}}{=} Q$, then $Q \overset{\text{val}}{=} P$; and
(3) if $P \overset{\text{val}}{=} Q$ and $Q \overset{\text{val}}{=} R$, then $P \overset{\text{val}}{=} R$

Substitution

Simple

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P] \stackrel{val}{=} \psi[\xi/P]}$$

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EVERY occurrence of P is substituted!

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Sequential

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P][\eta/Q] \stackrel{val}{=} \psi[\xi/P][\eta/Q]}$$

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Simultaneous

$$\frac{\phi \stackrel{val}{=} \psi}{\phi[\xi/P, \eta/Q] \stackrel{val}{=} \psi[\xi/P, \eta/Q]}$$

EVERY occurrence of P is substituted!

Substitution

meta rule

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EVERY occurrence of P is substituted!

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The rule of Leibniz

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$$\frac{\phi \overset{val}{=} \psi}{C[\phi] \overset{val}{=} C[\psi]}$$

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single occurrence is replaced!

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formula that has ϕ as a sub formula

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meta rule

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formula that has ϕ as a sub formula

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Strengthening and weakening

We had

Definition: Two abstract propositions P and Q are equivalent, notation $P \stackrel{val}{=} Q$, iff

(1) Always when P has truth value 1, also Q has truth value 1, and

(2) Always when Q has truth value 1, also P has truth value 1.

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if we relax this, we get strengthening

Strengthening

Definition: The abstract proposition P is stronger than Q, notation $P \models_{val} Q$, iff
~~(1)~~ Always when P has truth value 1, also Q has truth value 1~~, and~~
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Strengthening

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Q is weaker than P

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Lemma W4: $P \overset{val}{\models} Q$ iff $P \Rightarrow Q$ is a tautology.

Standard Weakenings

and-or-weakening

$$P \wedge Q \overset{val}{\models} P$$

$$P \overset{val}{\models} P \vee Q$$

Extremes

$$\mathrm{F} \overset{val}{\models} P$$

$$P \overset{val}{\models} \mathrm{T}$$

Calculating with weakenings
(the use of standard weakenings)

Substitution

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EVERY occurrence of P is substituted!

The rule of Leibnitz

does not hold for weakening!

Leibnitz

$$\cancel{\dfrac{\phi \overset{val}{\models} \psi}{C[\phi] \overset{val}{\models} C[\psi]}}$$

formula that has ϕ as a sub formula does not hold for weakening!

Leibnitz

$$\frac{\phi \overset{val}{\models} \psi}{C[\phi] \overset{val}{\models} C[\psi]}$$

Monotonicity

$$\frac{P \overset{val}{\models} Q}{P \wedge R \overset{val}{\models} Q \wedge R}$$

$$\frac{P \overset{val}{\models} Q}{P \vee R \overset{val}{\models} Q \vee R}$$