The structure of natural numbers

is helpful for proving properties $\forall n[n \in \mathbb{N}: P(n)]$

The structure of natural numbers

On natural numbers we can define a notion of a successor, a mapping

$$s: \mathbb{N} \to \mathbb{N}$$

by
$$s(n) = n+1$$

The successor mapping imposes a structure on the set that enables us to count:

- 1) there is a starting natural number 0
- 2) for every natural number n, there is a next natural number s(n) = n+1.

Important properties

(I) Different natural numbers have different successors:

$$\forall n,m [n,m \in \mathbb{N} : s(m) = s(n) \Rightarrow m = n]$$

Important properties

(I) Different natural numbers have different successors:

$$\forall n,m [n,m \in \mathbb{N} : s(m) = s(n) \Rightarrow m = n]$$

stated positively

Important properties

(I) Different natural numbers have different successors:

$$\forall n,m [n,m \in \mathbb{N} : s(m) = s(n) \Rightarrow m = n]$$

stated positively

s is injective!

Important properties

(I) Different natural numbers have different successors:

$$\forall n,m [n,m \in \mathbb{N} : s(m) = s(n) \Rightarrow m = n]$$

stated positively

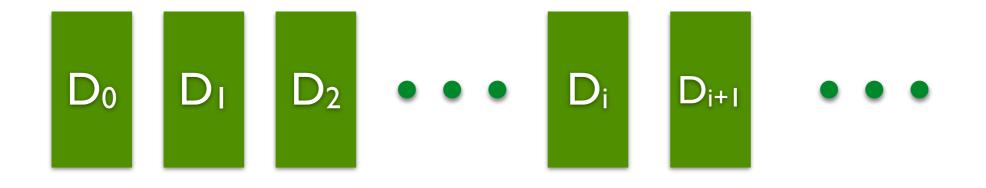
s is injective!

- (2) 0 is not a successor: $\forall n [n \in \mathbb{N} : \neg (s(n) = 0)]$
- (3) All natural numbers except 0 are successors:

$$\forall n[n \in \mathbb{N} \land \neg(n = 0) : \exists m[m \in \mathbb{N} : n = s(m)]$$

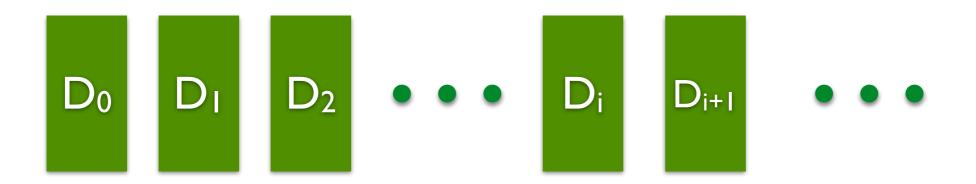
There is more to it - induction

Imagine an infinite sequence of dominos



There is more to it - induction

Imagine an infinite sequence of dominos



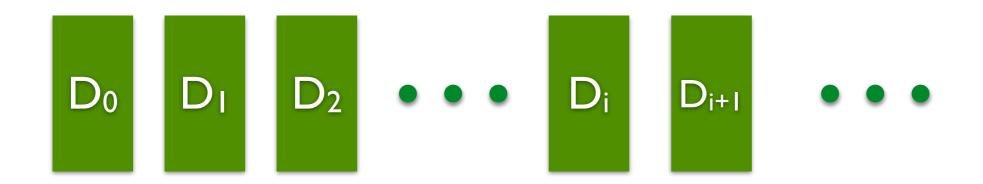
If we know that

- I. D_0 falls
- 2. The dominos are close enough together so that if D_i falls, then D_{i+1} falls (for all $i \in \mathbb{N}$)

Then we can conclude that every domino D_n ($n \in \mathbb{N}$) falls!

There is more to it - induction

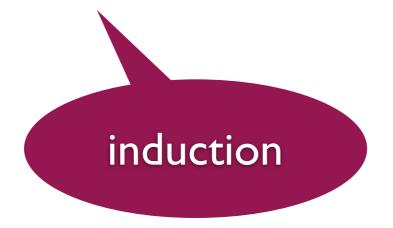
Imagine an infinite sequence of dominos



If we know that

- I. D_0 falls
- 2. The dominos are close enough together so that if D_i falls, then D_{i+1} falls (for all $i \in \mathbb{N}$)

Then we can conclude that every domino D_n ($n \in \mathbb{N}$) falls!



 $P(0) \land \forall i \ [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n \ [n \in \mathbb{N} : P(n)]$

P - unary predicate over N

 $P(0) \land \forall i \ [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n \ [n \in \mathbb{N} : P(n)]$

P - unary predicate over N

$$P(0) \land \forall i \ [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n \ [n \in \mathbb{N} : P(n)]$$

$$P(0)$$

$$P(0) \Rightarrow P(1)$$

$$P(1)$$

$$P(1) \Rightarrow P(2)$$

$$P(2)$$

$$P(2) \Rightarrow P(3)$$
...

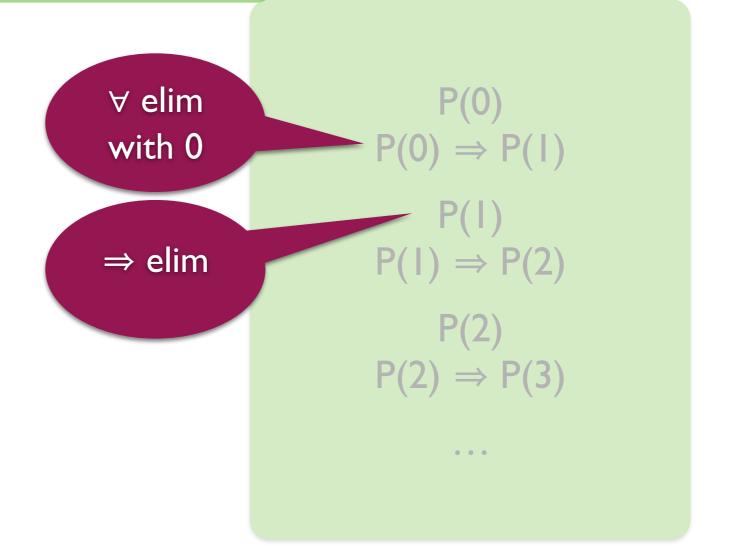
P - unary predicate over N

$$P(0) \land \forall i \ [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n \ [n \in \mathbb{N} : P(n)]$$

P(0)with 0 $P(0) \Rightarrow P(1)$ P(1) $P(1) \Rightarrow P(2)$ P(2) $P(2) \Rightarrow P(3)$...

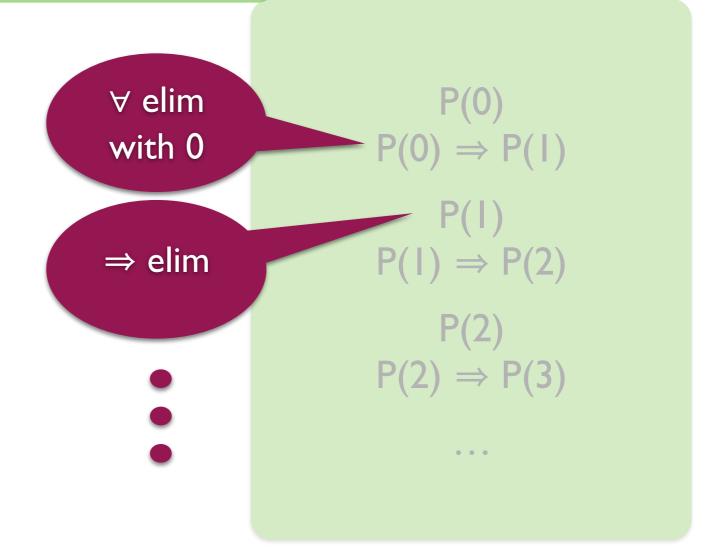
P - unary predicate over N

$$P(0) \land \forall i \ [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \ \Rightarrow \forall n \ [n \in \mathbb{N} : P(n)]$$



P - unary predicate over N

$$P(0) \land \forall i \ [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n \ [n \in \mathbb{N} : P(n)]$$



P - unary predicate over N

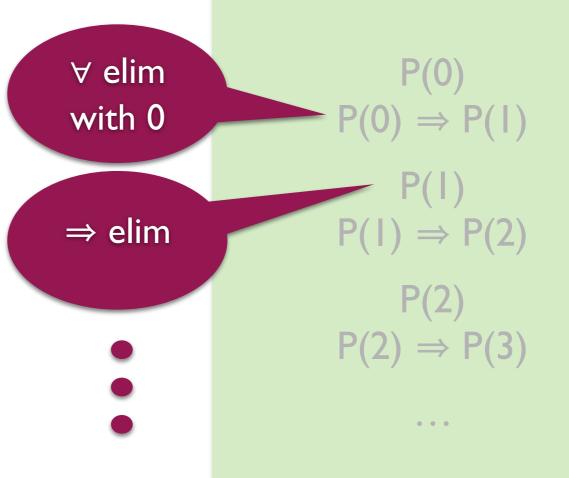
$$P(0) \land \forall i \ [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n \ [n \in \mathbb{N} : P(n)]$$

Variant of the Peano Axiom:

Let $K \subseteq \mathbb{N}$ have the property that

- (a) $0 \in K$ and
- (b) for all $n \in \mathbb{N}$, $n \in K \Rightarrow (n+1) \in K$.

Then $K = \mathbb{N}$.



 $P(0) \land \forall i \ [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n \ [n \in \mathbb{N} : P(n)]$

P - unary predicate over N

```
P(0) \, \wedge \, \forall i \; [i \in \mathbb{N}: \, P(i) \Rightarrow P(i+1)] \ \Rightarrow \forall n \; [n \in \mathbb{N}: \, P(n)]
```

P - unary predicate over N

```
P(0)
 (m)
        {Assume}
(k)
        var i; i \in \mathbb{N}
(k+1)
         \{\Rightarrow-intro on (k+1) and (I-1)\}
        P(i) \Rightarrow P(i+1)
        \{\forall-intro on (k) and (l)\}
(I+I) \forall i[i \in \mathbb{N} : P(i) \Rightarrow P(i+I)]
        {induction on (m) and (I+I)}
(I+2) \foralln[n \in N : P(n)]
```

```
P(0) \, \wedge \, \forall i \; [i \in \mathbb{N}: \, P(i) \Rightarrow P(i+1)] \ \Rightarrow \forall n \; [n \in \mathbb{N}: \, P(n)]
```

P - unary predicate over N

```
(m)
         {Assume}
(k)
         \mathbf{var} \; \mathbf{i}; \mathbf{i} \in \mathbb{N}
(k+1)
          P(i+1)
          \{\Rightarrow-intro on (k+1) and (I-1)\}
        | P(i) \Rightarrow P(i+1)
         \{\forall-intro on (k) and (l)\}
(I+I) \forall i[i \in \mathbb{N} : P(i) \Rightarrow P(i+I)]
         {induction on (m) and (I+I)}
(I+2) \foralln[n \in N : P(n)]
```

Basis

```
P(0) \wedge \forall i \ [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \ \Rightarrow \forall n \ [n \in \mathbb{N} : P(n)]
```

P - unary predicate over N

```
(m)
         {Assume}
(k)
        var i; i \in \mathbb{N}
(k+1)
         | P(i+1)
         \{\Rightarrow-intro on (k+1) and (I-1)\}
(l)
        | P(i) \Rightarrow P(i+1)
        \{\forall-intro on (k) and (l)\}
(I+I) \ \forall i[i \in \mathbb{N} : P(i) \Rightarrow P(i+I)]
        {induction on (m) and (I+I)}
(I+2) \foralln[n \in N : P(n)]
```

Basis

Induction step

```
P(0) \wedge \forall i \ [i \in \mathbb{N}: \ P(i) \Rightarrow P(i+1)] \ \Rightarrow \forall n \ [n \in \mathbb{N}: \ P(n)]
```

P - unary predicate over N

```
(m)
         {Assume}
(k)
         var i; i \in \mathbb{N}
(k+1)
(I-I)
          P(i+1)
         \{\Rightarrow-intro on (k+1) and (l-1)\}
        | P(i) \Rightarrow P(i+1)
(l)
        \{\forall-intro on (k) and (l)\}
(I+I) \forall i[i \in \mathbb{N} : P(i) \Rightarrow P(i+I)]
         {induction on (m) and (I+I)}
(I+2) \foralln[n \in N : P(n)]
```

Basis

induction hypothesis

Induction step

Inductive proof: truth is passed on

Inductive definition: construction is passed on

Inductive proof: truth is passed on

Inductive definition: construction is passed on

well defined by induction

Inductive proof: truth is passed on

Inductive definition: construction is passed on

well defined by induction

Example

The sequence of real numbers $(a_i \mid i \in \mathbb{N})$ is defined inductively by

$$a_0 = 2$$

 $a_{i+1} = 2a_i - 1$

Inductive proof: truth is passed on

Inductive definition: construction is passed on

well defined by induction

Example

The sequence of real numbers $(a_i \mid i \in \mathbb{N})$ is defined inductively by

$$a_0 = 2$$

 $a_{i+1} = 2a_i - 1$

a ₀	aı	a ₂	a ₃	a4	• • •
2	3	5	9	17	

Inductive proof: truth is passed on

Inductive definition: construction is passed on

well defined by induction

Example

The sequence of real numbers $(a_i \mid i \in \mathbb{N})$ is defined inductively by

$$a_0 = 2$$

 $a_{i+1} = 2a_i - 1$

a ₀	aı	a ₂	аз	a4	• • •
2	3	5	9	17	

Conjecture

For all $n \in \mathbb{N}$ it holds that

$$a_n = 2^n + 1$$

Inductive proof: truth is passed on

well defined by induction

Inductive definition: construction is passed on

Example

The sequence of real numbers $(a_i \mid i \in \mathbb{N})$ is

defined inductively by

$$a_0 = 2$$

 $a_{i+1} = 2a_i - 1$

a ₀	aı	a ₂	a ₃	a4	• • •
2	3	5	9	17	

proof by induction

Conjecture

For all $n \in \mathbb{N}$ it holds that

$$a_n = 2^n + 1$$

P - unary predicate over N

 $\forall k \ [k \in \mathbb{N} : \forall j [j \in \mathbb{N} \ \land \ j \le k : P(j)] \Rightarrow P(k)] \ \Rightarrow \forall n \ [n \in \mathbb{N} : P(n)]$

P - unary predicate over N

 $\forall k \ [k \in \mathbb{N} : \forall j [j \in \mathbb{N} \land j \le k : P(j)] \Rightarrow P(k)] \Rightarrow \forall n \ [n \in \mathbb{N} : P(n)]$

$$P(0)$$

$$P(0) \Rightarrow P(1)$$

$$P(0) \land P(1)$$

$$P(0) \land P(1) \Rightarrow P(2)$$

$$P(0) \land P(1) \land P(2)$$

$$P(0) \land P(1) \land P(2) \Rightarrow P(3)$$
...

P - unary predicate over N

```
\forall k \; [k \in \mathbb{N}: \; \forall j [j \in \mathbb{N} \; \land \; j \leq k : P(j)] \Rightarrow P(k)] \; \Rightarrow \forall n \; [n \in \mathbb{N}: \; P(n)]
```

 \forall elim with k=1

$$P(0)$$

$$P(0) \Rightarrow P(1)$$

$$P(0) \land P(1)$$

$$P(0) \land P(1) \Rightarrow P(2)$$

$$P(0) \land P(1) \land P(2)$$

$$P(0) \land P(1) \land P(2) \Rightarrow P(3)$$
...

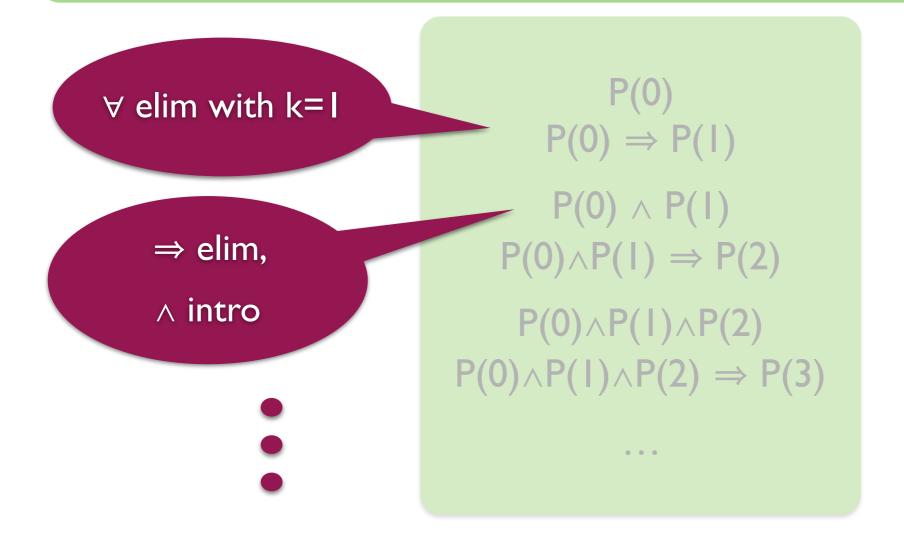
P - unary predicate over N

 $\forall k \ [k \in \mathbb{N} : \forall j [j \in \mathbb{N} \land j \le k : P(j)] \Rightarrow P(k)] \Rightarrow \forall n \ [n \in \mathbb{N} : P(n)]$

 $\begin{array}{c} \forall \text{ elim with } k=1 \\ P(0) \\ P(0) \Rightarrow P(1) \\ \hline P(0) \land P(1) \\ P(0) \land P(1) \Rightarrow P(2) \\ \hline P(0) \land P(1) \land P(2) \\ \hline P(0) \land P(1) \land P(2) \Rightarrow P(3) \\ \hline \end{array}$

P - unary predicate over N

 $\forall k \; [k \in \mathbb{N}: \; \forall j [j \in \mathbb{N} \; \land \; j \leq k : P(j)] \Rightarrow P(k)] \; \Rightarrow \forall n \; [n \in \mathbb{N}: \; P(n)]$



P - unary predicate over N

 $\forall k \; [k \in \mathbb{N}: \; \forall j [j \in \mathbb{N} \; \land \; j \leq k : P(j)] \Rightarrow P(k)] \; \Rightarrow \forall n \; [n \in \mathbb{N}: \; P(n)]$

 $\begin{array}{c} \forall \text{ elim with } k=1 \\ & P(0) \\ & P(0) \Rightarrow P(1) \\ \hline \\ & P(0) \land P(1) \\ & P(0) \land P(1) \Rightarrow P(2) \\ & P(0) \land P(1) \land P(2) \\ & P(0) \land P(1) \land P(2) \Rightarrow P(3) \\ \hline \\ & \cdots \\ & \end{array}$

Definition of $(a_i \mid i \in \mathbb{N})$ with strong induction

 a_n is defined via $a_0, ..., a_{n-1}$