

The structure of natural numbers

is helpful for proving
properties
 $\forall n[n \in \mathbb{N} : P(n)]$

The structure of natural numbers

On natural numbers we can define a notion of a **successor**, a mapping

$$s: \mathbb{N} \rightarrow \mathbb{N}$$

by $s(n) = n+1$

The successor mapping imposes a structure on the set that enables us to **count**:

- 1) there is a **starting** natural number 0
- 2) for every natural number n , there is a **next** natural number $s(n) = n+1$.

(Some) Peano Axioms

Important properties

(I) Different natural numbers have different successors:

$$\forall n, m [n, m \in \mathbb{N} : s(m) = s(n) \Rightarrow m = n]$$

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(2) 0 is not a successor: $\forall n [n \in \mathbb{N} : \neg (s(n) = 0)]$

(3) All natural numbers except 0 are successors:

$$\forall n [n \in \mathbb{N} \wedge \neg(n = 0) : \exists m [m \in \mathbb{N} : n = s(m)]]$$

There is more to it - induction

Imagine an infinite sequence of dominos



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If we know that

1. D_0 falls
2. The dominos are close enough together so that if D_i falls, then D_{i+1} falls (for all $i \in \mathbb{N}$)

Then we can conclude that every domino D_n ($n \in \mathbb{N}$) falls!

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induction

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$$P(0) \wedge \forall i [i \in \mathbb{N} : P(i) \Rightarrow P(i+1)] \Rightarrow \forall n [n \in \mathbb{N} : P(n)]$$

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over \mathbb{N}

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Variant of the Peano Axiom:

Let $K \subseteq \mathbb{N}$ have the property that

(a) $0 \in K$ and

(b) for all $n \in \mathbb{N}$, $n \in K \Rightarrow (n+1) \in K$.

Then $K = \mathbb{N}$.

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{Assume}

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(l-1) P(i+1)
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hypothesis

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Inductive proof: truth is passed on

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Example

The sequence of real numbers $(a_i \mid i \in \mathbb{N})$ is defined inductively by

$$a_0 = 2$$

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2	3	5	9	17	...

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For all $n \in \mathbb{N}$ it holds that

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Definition of
 $(a_i \mid i \in \mathbb{N})$
with strong
induction

a_n is defined via
 a_0, \dots, a_{n-1}