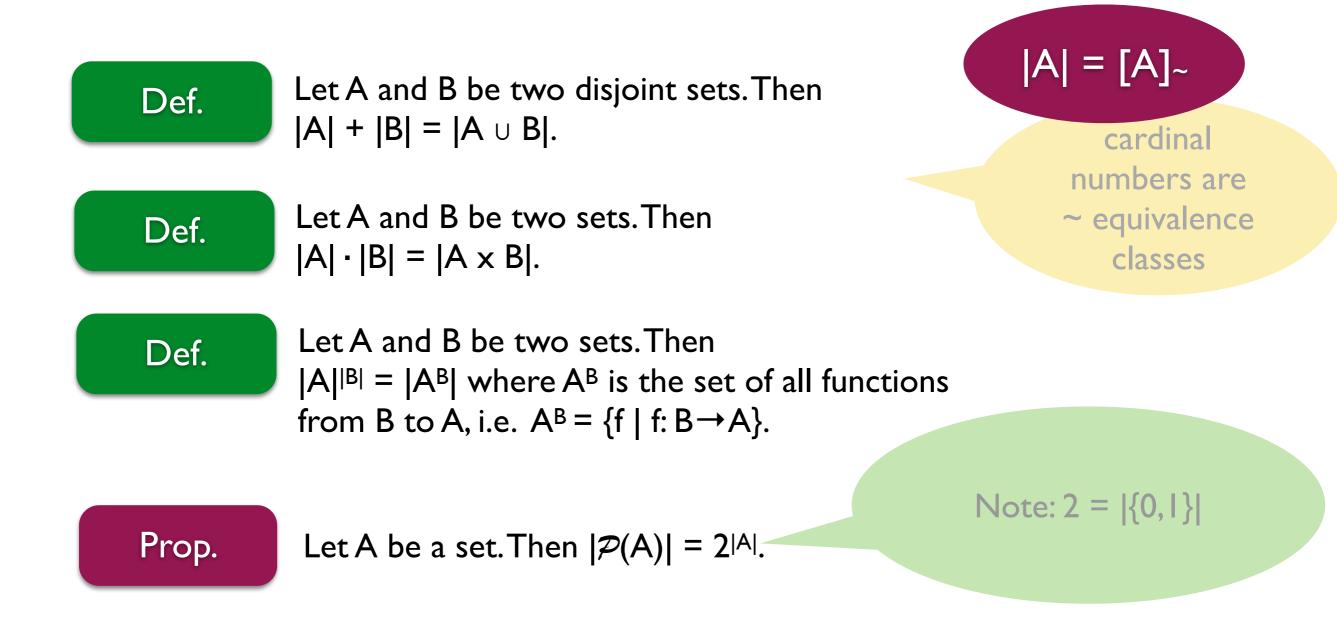
Cardinality

### Cardinals

Def.	Two sets A and B have the same cardinality (are equinumerous) if there is a bijection $f: A \rightarrow B$ . Notation A ~ B, or $ A  =  B $ .	A  = [A]~
Prop.	The relation $\sim$ is an equivalence relation on sets.	cardinal numbers are ~ equivalence
Def.	A set A has at most as large cardinality as a set B if there is an injection $f: A \rightarrow B$ . Notation $ A  \leq  B $ .	classes
Def.	A set A has at least as large cardinality as a set B if there is a surjection f:A $\rightarrow$ B. Notation  A  $\geq$  B .	Theorem (Cantor) If $ A  \le  B $
Def.	A set A has smaller cardinality than a set B if there is an injection $f:A \rightarrow B$ and there is no surjection $f:A \rightarrow B$ . Notation $ A  <  B $ .	and $ B  \leq  A ,$ then  A  =  B .

### **Operations on cardinals**



### Finite sets, finite cardinals

We write  $\mathbb{N}_k$  for the set  $\{0, 1, ..., k-1\}$ . Then  $\mathbb{N}_0 = \emptyset$ .

We will also write k for  $|N_k|$ .



A set A is finite if and only if  $|A| = |N_k|$ , for some  $k \in \mathbb{N}$ . We write then |A| = k.

Hence

A set A is finite if and only if there is a natural number  $k \in \mathbb{N}$  and a bijection f:A  $\rightarrow \mathbb{N}_{k}$ .

if and only if A has k elements, for some  $k \in \mathbb{N}$ 

E.g. If |A| = k and |B| = mfor some k,m  $\in \mathbb{N}$ then  $|AxB| = k \cdot m$ 

|A| = [A]~

cardinal

numbers are

 $\sim$  equivalence

classes

The operations on cardinals when restricted to finite cardinals coincide with the operations on natural numbers! This justifies the notation.

## Infinite, countable and uncountable sets

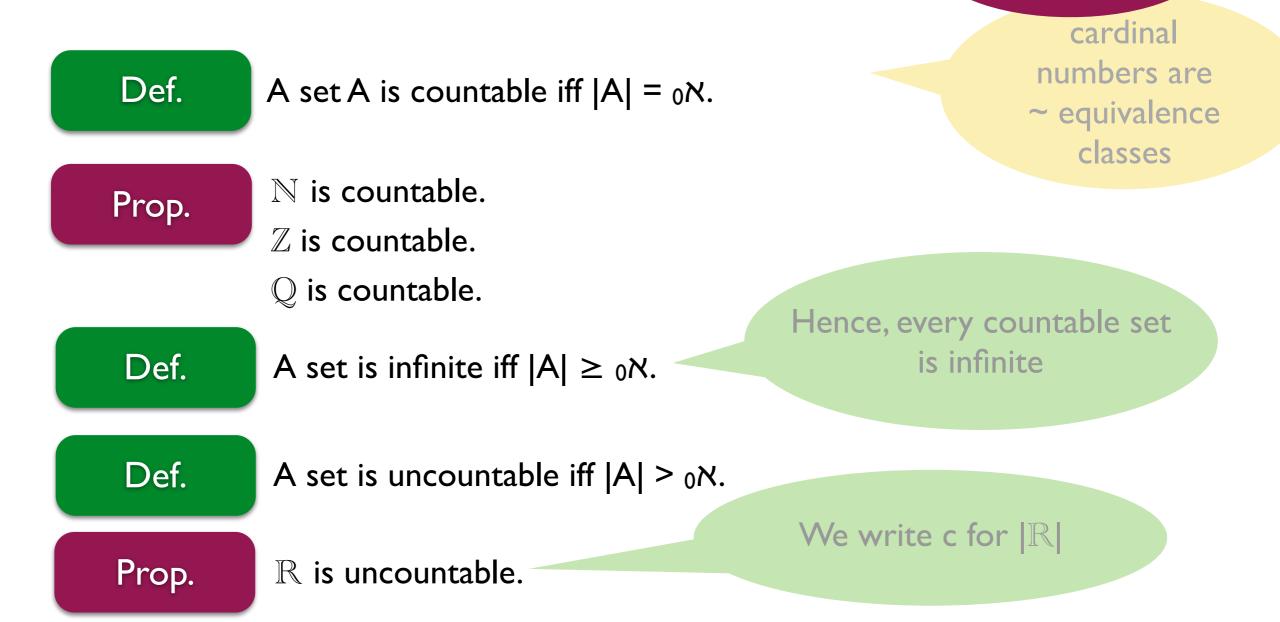
Time for a video!

Hilbert's infinite hotel :-)

# Infinite, countable and uncountable sets

|A| = [A]~

We write  $_{0}$  for the cardinality of natural numbers. Hence  $_{0}$  =  $|\mathbb{N}|$ .



#### Cardinals are unbounded

Theorem (Cantor)

For every set A we have  $|A| < |\mathcal{P}(A)|$ .

|A| = [A]~

cardinal numbers are ~ equivalence classes

Hence, for every cardinal there is a larger one.