# Cardinality

# Cardinals

**Def.** Two sets $A$ and $B$ have the same cardinality (are equinumerous) if there is a bijection $f: A \to B$. Notation $A \sim B$, or $|A| = |B|$.

$|A| = [A]_\sim$

cardinal numbers are $\sim$ equivalence classes

**Prop.** The relation $\sim$ is an equivalence relation on sets.

**Def.** A set $A$ has at most as large cardinality as a set $B$ if there is an injection $f: A \to B$. Notation $|A| \leq |B|$.

**Def.** A set $A$ has at least as large cardinality as a set $B$ if there is a surjection $f: A \to B$. Notation $|A| \geq |B|$.

**Def.** A set $A$ has smaller cardinality than a set $B$ if there is an injection $f: A \to B$ and there is no surjection $f: A \to B$. Notation $|A| < |B|$.

**Theorem (Cantor)**

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

# Operations on cardinals

$|A| = [A]_{\sim}$

cardinal numbers are $\sim$ equivalence classes

**Def.** Let A and B be two disjoint sets. Then
$|A| + |B| = |A \cup B|$.

**Def.** Let A and B be two sets. Then
$|A| \cdot |B| = |A \times B|$.

**Def.** Let A and B be two sets. Then
$|A|^{|B|} = |A^B|$ where $A^B$ is the set of all functions from B to A, i.e. $A^B = \{f \mid f: B \rightarrow A\}$.

**Prop.** Let A be a set. Then $|\mathcal{P}(A)| = 2^{|A|}$.

Note: $2 = |\{0,1\}|$

# Finite sets, finite cardinals

We write $\mathbb{N}_k$ for the set $\{0,1,\ldots,k-1\}$. Then $\mathbb{N}_0 = \varnothing$.

We will also write k for $|\mathbb{N}_k|$.

$|A| = [A]_\sim$

cardinal numbers are ~ equivalence classes

**Def.** A set A is finite if and only if $|A| = |\mathbb{N}_k|$, for some $k \in \mathbb{N}$. We write then $|A| = k$.

**Hence** A set A is finite if and only if there is a natural number $k \in \mathbb{N}$ and a bijection $f: A \to \mathbb{N}_k$.

if and only if A has k elements, for some $k \in \mathbb{N}$

E.g. If $|A| = k$ and $|B| = m$ for some $k,m \in \mathbb{N}$

then $|A \times B| = k \cdot m$

The operations on cardinals when restricted to finite cardinals coincide with the operations on natural numbers!
This justifies the notation.

# Infinite, countable and uncountable sets

Time for a video!

Hilbert's
infinite hotel :-)

# Infinite, countable and uncountable sets

We write $\aleph_0$ for the cardinality of natural numbers.
Hence $\aleph_0 = |\mathbb{N}|$.

$|A| = [A]_{\sim}$

cardinal numbers are ~ equivalence classes

**Def.** A set A is countable iff $|A| = \aleph_0$.

**Prop.** $\mathbb{N}$ is countable.
$\mathbb{Z}$ is countable.
$\mathbb{Q}$ is countable.

**Def.** A set is infinite iff $|A| \geq \aleph_0$.

Hence, every countable set is infinite

**Def.** A set is uncountable iff $|A| > \aleph_0$.

**Prop.** $\mathbb{R}$ is uncountable.

We write c for $|\mathbb{R}|$

# Cardinals are unbounded

$|A| = [A]_\sim$

cardinal numbers are $\sim$ equivalence classes

**Theorem (Cantor)**

For every set A we have $|A| < |\mathcal{P}(A)|$.

Hence, for every cardinal there is a larger one.