Finite Automata

Alphabets and Languages

Def

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Σ - alphabet (finite set)

$\Sigma^n = \{a_1a_2..a_n \mid a_i \in \Sigma\}$ is the set of words of length n

$\Sigma^* = \{w \mid \exists n \in \mathbb{N}.\ \exists\ a_1, a_2, .., a_n \in \Sigma.\ w = a_1a_2..a_n\}$ is the set of all words over Σ

Alphabets and Languages

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$\Sigma^0 = \{\mathcal{E}\}$ contains only the empty word

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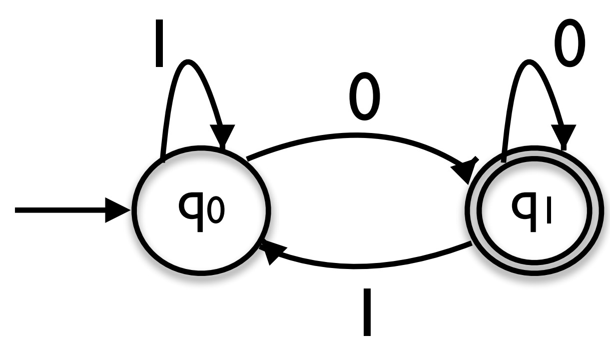
A language L over Σ is a subset $L \subseteq \Sigma^*$

Deterministic Automata (DFA)

Informal example

$\Sigma = \{0,1\}$

M_1:



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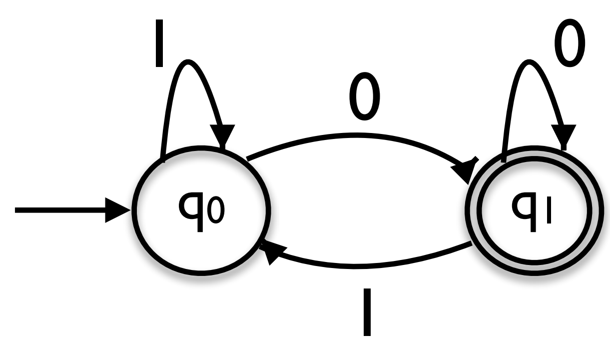

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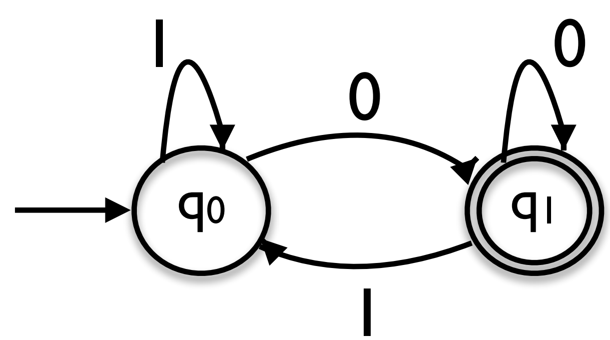

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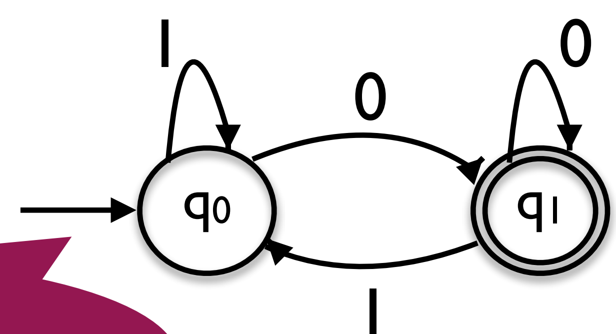

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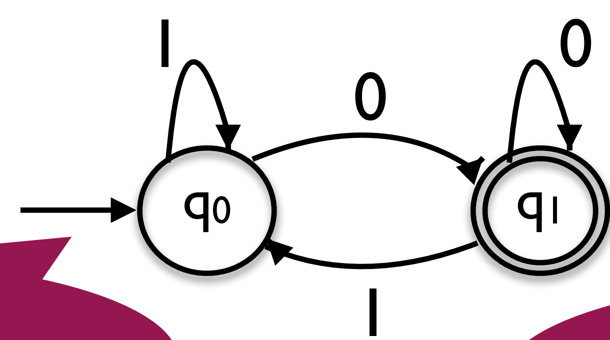

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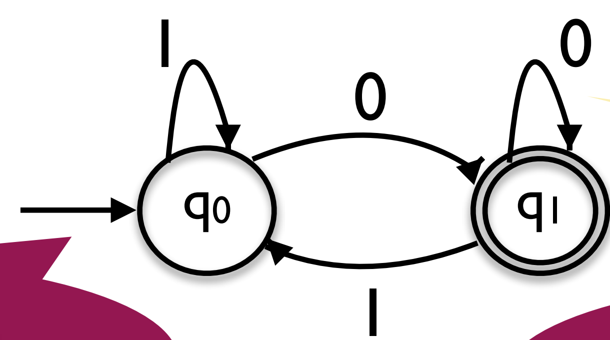

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transitions, labelled by alphabet symbols

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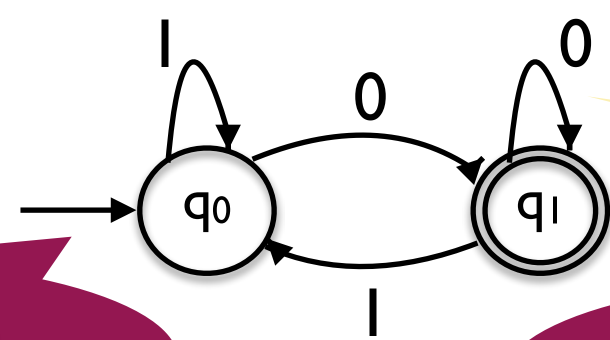

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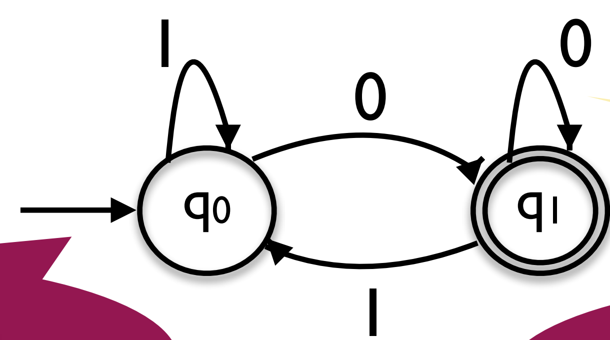

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regular language

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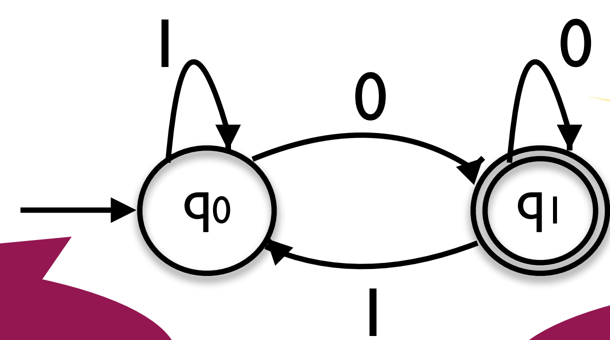

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regular expression

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$\delta(q_0, 0) = q_1$, $\delta(q_0, 1) = q_0$
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DFA

The extended transition function

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inductively, by:

$\delta^*(q, \varepsilon) = q$ and $\delta^*(q,wa) = \delta(\delta^*(q,w), a)$

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The language recognised / accepted by a deterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ is

$$L(M) = \{w \in \Sigma^* \mid \delta^*(q_0, w) \in F\}$$

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Regular languages and operations

Definition

Let Σ be an alphabet. A language L over Σ ($L \subseteq \Sigma^*$) is regular iff it is recognised by a DFA.

Regular languages and operations

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Regular operations

Let L, L_1, L_2 be languages over Σ. Then
$L_1 \cup L_2$, $L_1 \cdot L_2$, and L^* are languages, where

$$L_1 \cdot L_2 = \{w_1 \cdot w_2 \mid w_1 \in L_1, w_2 \in L_2\}$$

$$L^* = \{w \mid \exists n \in \mathbb{N}.\ \exists\, w_1, w_2, .., w_n \in L.\ w = w_1 w_2 .. w_n\}$$

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$\varepsilon \in L^*$ always

Closure under regular operations

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Theorem C1

The class of regular languages is closed under union

Closure under regular operations

also under intersection

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Theorem C2

The class of regular languages is closed under complement

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The class of regular languages is closed under union

Theorem C2

The class of regular languages is closed under complement

Theorem C3

The class of regular languages is closed under concatenation

Closure under regular operations

also under intersection

Theorem C1

The class of regular languages is closed under union

Theorem C2

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The class of regular languages is closed under concatenation

Theorem C4

The class of regular languages is closed under Kleene star

Closure under regular operations

also under intersection

We can already prove these!

Theorem C1

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Theorem C2

The class of regular languages is closed under complement

Theorem C3

The class of regular languages is closed under concatenation

But not yet these two…

Theorem C4

The class of regular languages is closed under Kleene star

Regular expressions

Definition

finite representation of infinite languages

Regular expressions

Definition

finite representation of infinite languages

Regular expressions

Definition

Let Σ be an alphabet. The following are regular expressions

1. a for $a \in \Sigma$
2. ε
3. $\emptyset$
4. $(R_1 \cup R_2)$ for R_1, R_2 regular expressions
5. $(R_1 \cdot R_2)$ for R_1, R_2 regular expressions
6. $(R_1)^*$ for R_1 regular expression

finite representation of infinite languages

Regular expressions

inductive

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finite representation of infinite languages

Regular expressions

inductive

example:
$(ab \cup a)^*$

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finite representation of infinite languages

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corresponding languages

$L(a) = \{a\}$

$L(\varepsilon) = \{\varepsilon\}$

$L(\emptyset) = \emptyset$

$L(R_1 \cup R_2) = L(R_1) \cup L(R_2)$

$L(R_1 \cdot R_2) = L(R_1) \cdot L(R_2)$

$L(R_1^*) = L(R_1)^*$

Equivalence of regular expressions and regular languages

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A language is regular (i.e., recognised by a finite automaton) iff it is the language of a regular expression.

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needs nondeterminism

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Nondeterministic Automata (NFA)

Informal example

$\Sigma = \{0,1\}$

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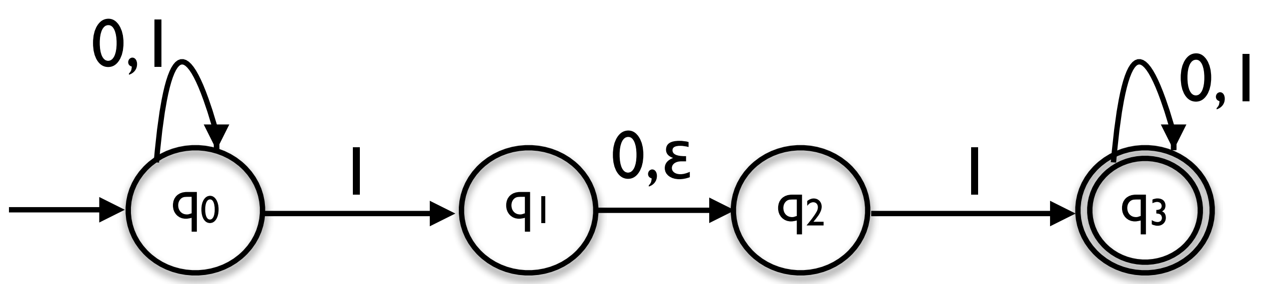

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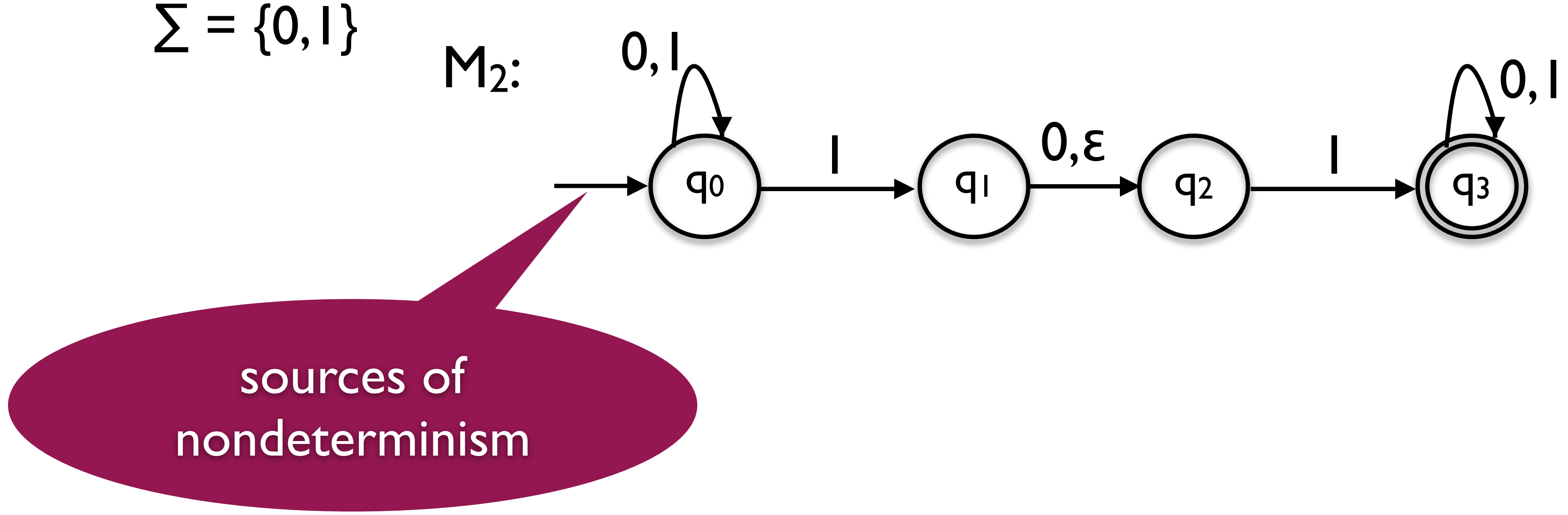

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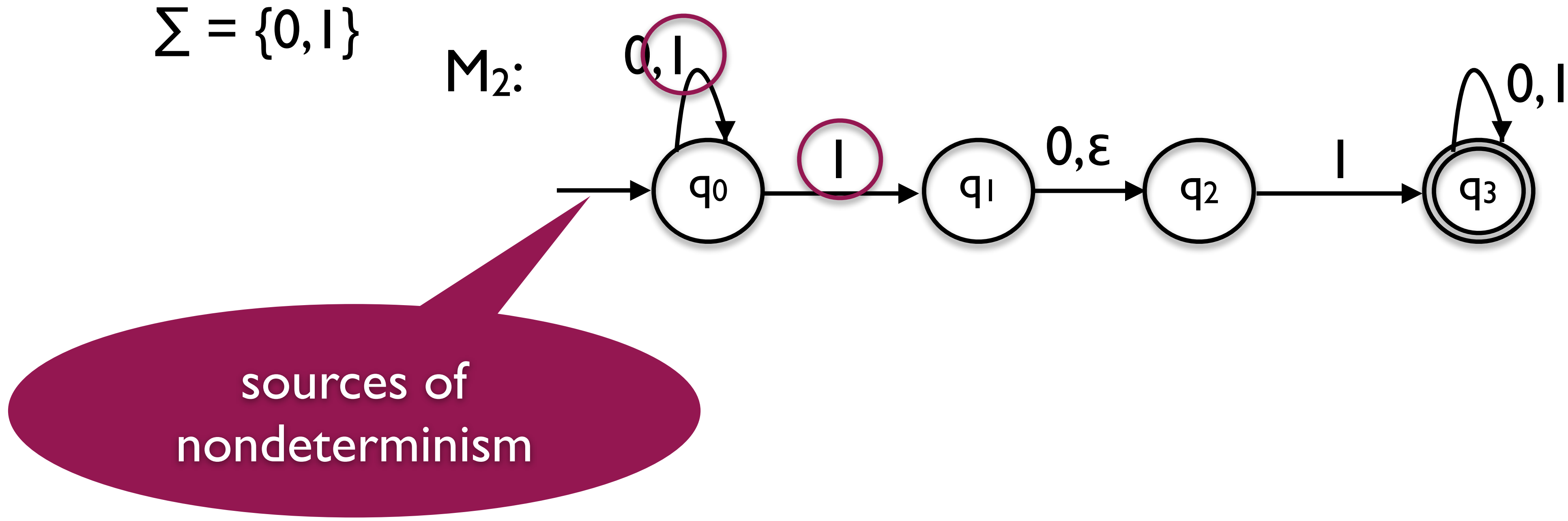

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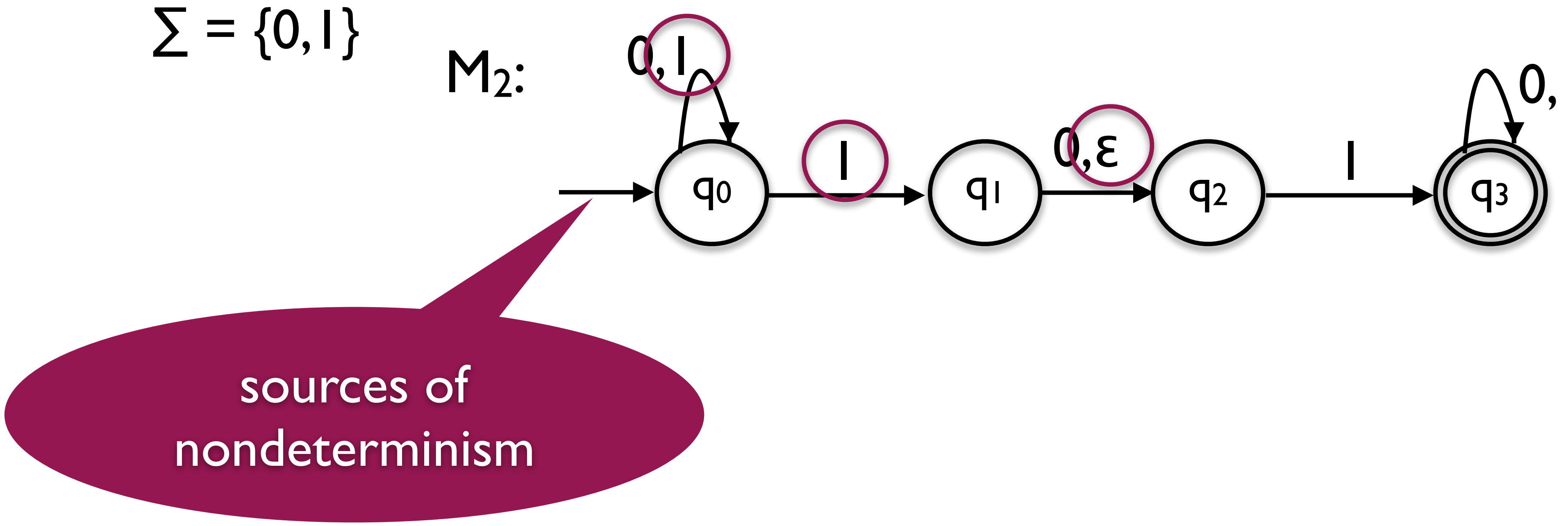

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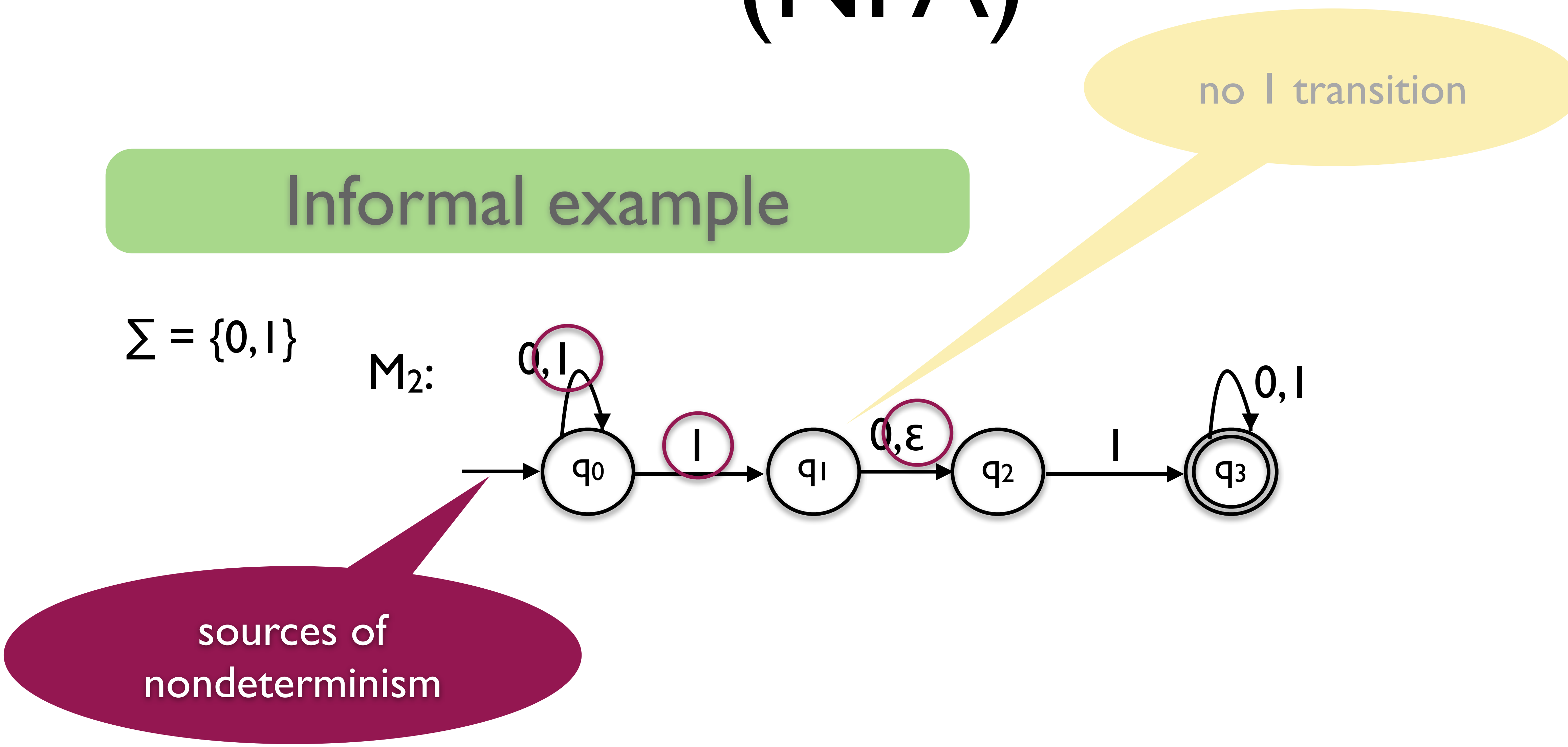

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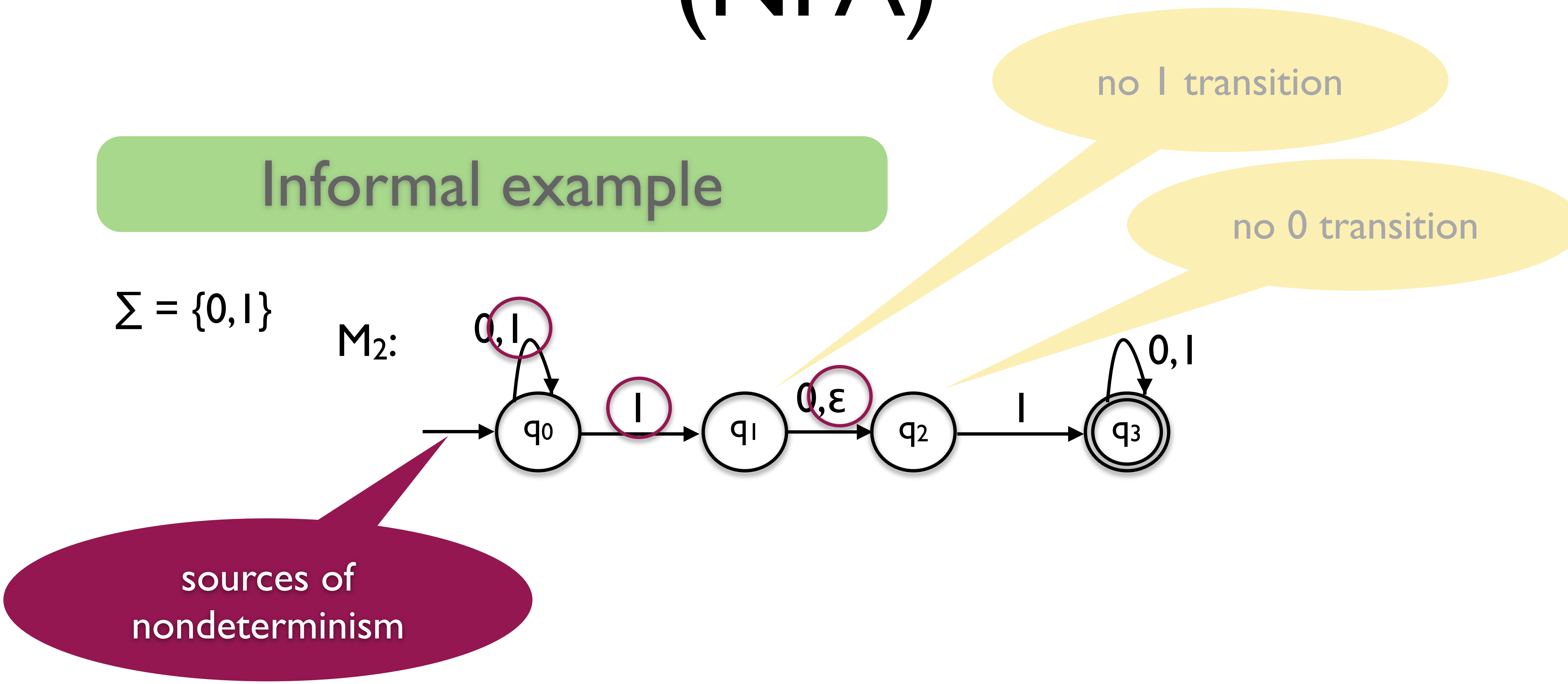

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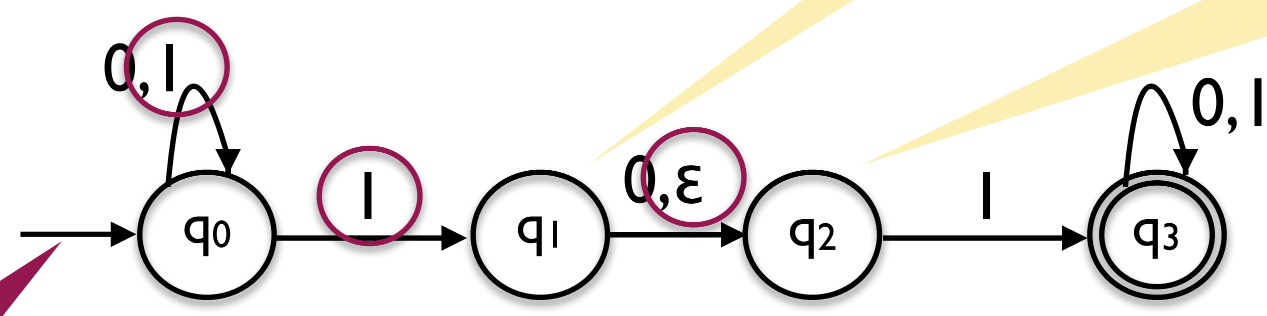

no 1 transition

no 0 transition

sources of nondeterminism

Accepts a word iff there exists an accepting run

NFA

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$\Sigma = \{0, 1\}$ $F = \{q_3\}$

$\delta(q_0, 0) = \{q_0\}$
$\delta(q_0, 1) = \{q_0, q_1\}$
$\delta(q_0, \varepsilon) = \varnothing$
.....

NFA

The extended transition function

NFA

The extended transition function

Given an NFA $M = (Q, \Sigma, \delta, q_0, F)$ we can extend $\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ to

$\delta^*: Q \times \Sigma^* \longrightarrow \mathcal{P}(Q)$

inductively, by:

$\delta^*(q, \varepsilon) = E(q)$ and $\delta^*(q,wa) = E(\bigcup_{q' \in \delta^*(q,w)} \delta(q', a))$

NFA

$E(q) = \{q' \mid q' = q \vee \exists n \in \mathbb{N}^{+}. \exists q_0, .., q_n \in Q. q_0 = q, q_n = q', q_{i+1} \in \delta(q_i, \varepsilon), \text{ for } i = 0, .., n-1\}$

The extended transition function

Given an NFA $M = (Q, \Sigma, \delta, q_0, F)$ we can extend $\delta: Q \times \Sigma_\varepsilon \longrightarrow \mathcal{P}(Q)$ to

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inductively, by:

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ε-closure of q, all states reachable by ε-transitions from q

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Definition

The language recognised / accepted by a nondeterministic finite automaton $M = (Q, \Sigma, \delta, q_0, F)$ is

$$L(M) = \{w \in \Sigma^* \mid \delta^*(q_0, w) \cap F \neq \emptyset\}$$

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$L(M_2) = \{u101w \mid u,w \in \{0,1\}^*\}$
$\cup$
$\{u11w \mid u,w \in \{0,1\}^*\}$

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Corollary

A language is regular iff it is recognised by a NFA

Closure under regular operations

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Theorem C1

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Now we can prove these too

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every long enough word of a
regular language can be pumped

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every long enough word of a regular language can be pumped

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If L is a regular language, then there is a number $p \in \mathbb{N}$ (the pumping length) such that for any $w \in L$ with $|w| \geq p$, there exist $x, y, z \in \Sigma^*$ such that $w = xyz$ and

1. $xy^iz \in L$, for all $i \in \mathbb{N}$
2. $|y| > 0$
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$L = \{ 0^n1^n \mid n \in \mathbb{N}\}$ is nonregular.

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Note the logical structure!